

Information, Contract Enforcement, and Misallocation

JOB MARKET PAPER

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Abstract

Misallocation of resources can cause large reductions in total factor productivity (TFP). The literature emphasizes financial frictions driven by limited contract enforcement that restrict productive firms' access to credit. Evidence suggests that information frictions also reduce access to credit, particularly in countries with weak contract enforcement. I study how the interaction between information frictions and limited enforcement affects resource allocation and TFP. I build a model in which lenders have imperfect information about borrowers' default risk and enforcing repayment is costly. I use the model to illustrate i) how imperfect information of this type causes misallocation, and ii) how limited enforcement exacerbates this effect. I calibrate the model and find that imperfect information causes TFP to fall by up to 23% when I take contract enforcement parameter values from U.S. data, and by up to 32% when I set them to values common in low-income countries.

Keywords: Misallocation, TFP, financial frictions, information frictions, limited enforcement, default

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1 Introduction

Misallocation of resources across firms or establishments is an important source of variation in total factor productivity across countries. (Banerjee and Duflo, 2005; Restuccia and Rogerson, 2008; Hsieh and Klenow, 2009). The development literature proposes a variety of potential sources of misallocation but financial frictions receive the most attention. The basic idea is that in countries with poorly-functioning financial markets, productive firms are often constrained from borrowing enough to reach their optimal sizes. Several recent quantitative studies like Buera, Kaboski, and Shin (2011) and Amaral and Quintin (2010) have found that can financial frictions can explain a significant fraction of cross-country variation in TFP. The bulk of the literature that studies the quantitative impact of financial frictions on misallocation and TFP emphasizes limited contract enforcement as the source of these frictions, abstracting from other sources of cross-country differences in financial development. Empirical evidence indicates that depth of credit information – existence of public or private credit bureaus and the depth of their coverage – is associated with increased credit to the private sector, and that this effect is more pronounced in countries with weaker contract enforcement (Jappelli and Pagano, 2002; Djankov, McLiesh, and Schleifer, 2007; Brown, Jappelli, and Pagano, 2009). This suggests that information about borrowers has a larger impact on lending when contracts are harder to enforce. In this paper I study how imperfect information in financial markets affects resource allocation and TFP, and how this effect changes with the strength of contract enforcement.

Building on the results cited above, I use data from the World Bank’s *Doing Business* project and the Penn World Tables to study the empirical relationship across countries between depth of credit information¹ and TFP, and how the strength of contract enforcement affects this relationship. In section 2 I show that both depth of credit information and the cost of enforcing contracts are associated with higher TFP, and that the interaction between these two variables is negative and significant. These results indicate that a change in depth of credit information is associated with a larger change in TFP when contract enforcement is weak. This suggests that not only does information play a larger role in increasing lending in countries with weak contract enforcement as documented by the studies cited above, it also plays a larger role in allocating resources efficiently.

In the rest of the paper I study a model motivated by these results that illustrates how imperfect information about borrowers can cause misallocation and reduce TFP. Section 3 describes the simplest form of the model, in which lenders make uncontingent loans to borrowers who have different exogenous default probabilities. Two financial frictions affect the contract terms that lenders offer: limited enforcement and

¹As I describe below, this *Doing Business* depth of credit information measure is a ranking that captures the existence and breadth of coverage of public and private credit bureaus. It is constructed using a methodology based on Djankov, McLiesh, and Schleifer (2007).

imperfect information. I model limited enforcement as a cost that lenders must pay to recover from borrowers who default, and imperfect information as noisy signals about borrowers' types that creates uncertainty about borrowers' default risk. When these signals become more noisy, lenders form less accurate estimates of default risk. When contract enforcement is expensive lenders receive small payoffs when borrowers default, so default risk has a larger impact on the loan terms lenders offer. Conversely, when contract enforcement is cheap lenders can recover most of the loaned funds from defaulters, which reduces the importance of default risk. As a consequence, imperfect information, which affects the accuracy of lenders' estimates of borrowers' default risk, has a larger impact on lending when contract enforcement is weak. In section 4 I embed this information and contracting structure into a production economy with heterogeneous firms that must borrow to finance investment, and prove several analytical results that illustrate how imperfect information causes misallocation and how this effect is more pronounced when contract enforcement is more expensive. Compared to a perfect-information environment, productive firms for which lenders overestimate default risk face tight borrowing constraints and high interest rates, causing them to make smaller investments, while unproductive firms for which lenders underestimate default risk receive better loan terms and make larger investments. This leads to a less efficient allocation of capital and lower TFP.

Several studies in the misallocation literature like Buera and Shin (2011) and Moll (2012) argue that in a dynamic setting, firms' ability to save and build up capital over time using internal funds (*self-financing*) may mitigate the misallocative effects of financial frictions. The persistence of firm-level productivity, exit rates, adjustment costs, and other parameters that affect the efficacy of self-financing play a crucial role in determining the magnitude of this effect. In order to obtain a quantitative assessment of the combined impact of imperfect information and limited enforcement, we need a dynamic model that takes these forces into account. In section 5 I extend my model to a quantitative setting in which firms can build up capital over time. In this version of the model, firm-level productivity has two components, one persistent and one i.i.d. This allows me to calibrate the productivity process to match the stationary firm size distribution as well as the persistence and volatility of firm-level productivity. I calibrate the perfect-information, low-enforcement cost version of the model to the U.S. economy and study the effects of introducing information frictions and increasing the enforcement cost on TFP and GDP per capita. When I hold the enforcement cost to the baseline parameterization I find that imperfect information can reduce TFP by up to 23 percent. When I use contract enforcement parameter values from *Doing Business* for low-income countries (a similar exercise to that conducted by D'Erasmus and Boedo, 2012), imperfect information reduces TFP by up to 32 percent. In addition to misallocating capital across firms, the information and enforcement frictions in the dynamic model also reduce the aggregate capital stock. As a consequence, GDP per capita falls by up to 57

percent in the baseline calibration and by up to 67 percent in the weak enforcement scenario. These findings are robust to changes in productivity persistence, exit rates, and capital adjustment costs.

The remainder of the paper proceeds as follows. In section 2 I use cross-country data to analyze the empirical relationships across countries between depth of credit information, limited enforcement, and TFP. In section 3 I describe the financial market and information structure in my model. In section 4 I embed this framework in a production economy with heterogeneous firms and provide analytical results that illustrate how the two frictions in my model cause misallocation. In section 5 I present a dynamic version of the model, calibrate it, and assess the quantitative impact of imperfect information and limited enforcement on TFP in this richer environment. Section 6 concludes.

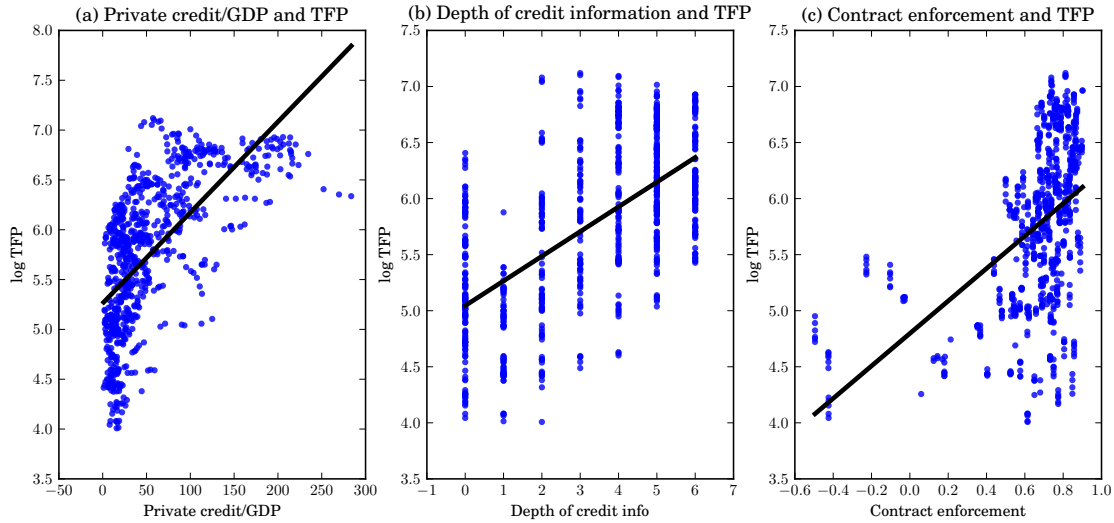
2 Empirical motivation

One of the reasons that the literature focuses on financial frictions as a source of misallocation is that there is a strong empirical connection between financial development and total factor productivity. Panel (a) of figure 1 plots total factor productivity against the typical measure of financial development, the ratio of credit to the private sector to gross domestic product. To construct my measure of TFP, I use data on population and purchasing-power adjusted real GDP and investment from the Penn World Tables version 7.1. I use the investment series to construct a capital stock series for each country in the dataset using the perpetual inventory method, then calculate TFP as a standard Solow residual: $TFP = Y / (K^{0.36} L^{0.64})$. Here Y is PPP real GDP, K is the capital stock and L is employment. The private credit/GDP ratio comes from the World Bank's World Development Indicators database. I only use country-year observations for which I also have data on depth of credit information and contract enforcement which I describe in more detail below. The year ranges from 2005 to 2010.

The quantitative literature on financial development and TFP typically interprets financial development (or lack thereof) in terms of the strength of contract enforcement. Limited enforcement is important, but it's not the whole story. Panels (b) and (c) of figure 1 illustrates this point. Panel (b) plots TFP against a measure of the depth of credit information, and panel (c) plots TFP against a measure of the strength of contract enforcement. All three panels depict strong positive relationships.

The depth of credit information and contract enforcement measures both come from the World Bank's *Doing Business* database, a project aimed at providing objective measures about business regulations, institutional quality and other aspects of the business environment for a wide range of countries. The depth of credit information index is a measure of coverage, scope and accessibility of information about individuals' and firms' credit histories. The index takes discrete values from 0 to 6. It is constructed by assigning one

Figure 1: Total factor productivity versus depth of credit information and contract enforcement



point each to the following features of a country's credit information environment:

- Both positive (e.g. amount of debt successfully repaid) and negative (e.g. late payments, number of defaults) credit information are distributed.
- Data on firms and individuals are distributed.
- Data from retailers and utility companies as well as financial institutions are distributed.
- More than 2 years of historical data are distributed and defaults are not erased on repayment.
- Data on loan amounts below 1 percent of income per capita are distributed.
- Borrowers have legal rights to access their credit histories.

Countries in which public or private credit bureaus do not exist or cover less than 1 percent of the adult population receive a zero on the depth of credit information index. The methodology used by the World Bank to construct the depth of credit index was developed by Djankov, McLiesh, and Schleifer (2007). The contract enforcement cost index is a measure of the cost, as a fraction of the claim, that a complainant will incur to enforce a typical contract. A higher value of this index means that contracts are more costly to enforce. The methodology used to construct this index was developed by Djankov, Porta, de Silane, and Shleifer (2002). Higher values of the depth of credit information are good rather than bad, so in order to be consistent I define my measure of a country's contract enforcement environment to be one minus the enforcement cost, i.e., the fraction of a claim that a claimant recoups after paying enforcement costs.

The scatter plots above suggest that both depth of credit information and contract enforcement are associated with higher TFP across countries. To demonstrate that the interaction between these two variables is also important for TFP I estimate the following regression:

$$\log TFP_{it} = \alpha + \beta_1 INFO_{it} + \beta_2 ENF_{it} + \beta_3 INFO_{it} * ENF_{it} + \gamma_i + \delta_t + u_{it} \quad (1)$$

The dependent variable is TFP in logs for country i at time t . The independent variables are $INFO_{it}$, the depth of credit index, ENF_{it} , my contract enforcement index, and an interaction term that captures the joint effect of these two variables. I include country and time fixed effects γ_i and δ_t to control for variation across countries and time in other factors that drive TFP. Table 1 below lists the results of this estimation. The

Table 1: The effects of depth of credit information and enforcement costs on TFP

Independent variable	(1)	(2)	(3)
Depth of credit information	0.013		0.030
	(0.003)‡		(0.008)‡
Contract enforcement		0.199	0.212
		(0.068) [†]	(0.077) [†]
Depth of credit info * enforcement			-0.025
			(0.011)*
Number of observations	988	861	854
Time and country fixed effects	Yes	Yes	Yes

Robust standard errors in parantheses. ‡, † and * indicate significance at the 0.1%, 1%, and 5% levels, respectively.

coefficients β_1 and β_2 on depth of credit information and contract enforcement are both positive and significant at the 0.1 percent and 1 percent levels respectively. This indicates that both of these dimensions of financial development are positively associated with TFP, even after controlling for unobserved heterogeneity across countries and time in other variables that also drive TFP. The coefficient β_3 on the interaction term is negative and significant. This indicates that TFP is more sensitive to changes in depth of credit information when enforcement costs are high, and more sensitive to changes in enforcement costs when depth of credit information is low. In other words, a reduction in the depth of credit information index is associated with a larger decrease in TFP in countries with high enforcement costs (equivalently, low values of the contract enforcement measure used in the regression). To get a sense of the economic magnitude

of these coefficients, recall that the depth of credit information index ranges from 0 to 6, and the contract enforcement index ranges from 0 to 1. Consider a country with a contract enforcement measure of zero (i.e. 100 percent enforcement costs). A one standard-deviation (2.19) drop in such a country's depth of credit information index is associated with a 6.6 percent drop in TFP. On the other hand, consider a country at the 90th percentile of contract enforcement (0.85). The same 2.19-point drop in the depth of credit information index in such a country is associated with a drop of only 1.9 percent

Put simply, these results suggest that high contract enforcement costs exacerbate the effects of poor depth of credit information on TFP. This is consistent with evidence documented by other studies that credit bureaus and other determinants of access to information about borrowers are associated with increased credit, especially in countries with weak contract enforcement (Jappelli and Pagano, 2002; Djankov, McLiesh, and Schleifer, 2007; Brown, Jappelli, and Pagano, 2009). Given these studies' findings and the high correlation between the credit to GDP ratio and TFP, my results are perhaps not particularly surprising. To my knowledge, however, no other studies have documented a direct relationship between depth of credit information and TFP. In the next section, I develop a model of misallocation driven by uncertainty about borrowers motivated by these findings.

3 Modeling imperfect information about default risk

In this section I describe how I model imperfect information about borrowers' default risk. Consider an economy populated by a large number of borrowers and lenders. Borrowers take out loans at interest rates set by lenders, who are risk-neutral and perfectly competitive, and therefore make zero profits in expectation from making these loans. I assume that markets are exogenously incomplete – these loans are not contingent upon any aggregate or idiosyncratic states, save for the fact that borrowers cannot commit to repay their loans. Each borrower has an exogenous probability of default that depends on the amount borrowed and the interest rate. Lenders have imperfect information about borrowers, however, and do not know borrowers' default probabilities with certainty.

In general terms, the key equation that determines the interest rate r on a loan of size ℓ is the following break-even condition, which states that the lender expects to make a profit of exactly zero on the loan:

$$(1 + r^*)\ell = (1 - \mathbb{E}[p(\ell, r)])(1 + r)\ell + \mathbb{E}[p(\ell, r)](1 - \phi)\ell \quad (2)$$

Here, $p(\ell, r)$ is the probability that the borrower defaults, which is a function of the loan size and the interest rate. This equation simply says that the lender must be indifferent between investing ℓ units of resources

at the risk-free rate r^* or making a risky loan to the borrower at an interest rate of r . If the borrower repays, the lender gets the loan back plus interest. If the borrower defaults, the lender gets a fraction $1 - \phi$ of the money advanced to the borrower. This is a standard condition in studies that model equilibrium default, appearing, for example, in the literatures on sovereign default (Aguiar and Gopinath, 2006; Arellano, 2008), consumer bankruptcy (Chatterjee, Corbae, Nakajima, and Ríos-Rull, 2007), and more recent studies on financial frictions and firm dynamics (D’Erasmus and Boedo, 2012; Arellano, Bai, and Zhang, 2011). The key difference in this paper is that the lender does not know the borrower’s default probability $p(\ell, r)$ with certainty; instead, she estimates it using the information available to her.² I use the expectation operator \mathbb{E} to express this uncertainty for now; I will be more precise about the process the lender uses to construct the estimate $\mathbb{E}[p(\ell, r)]$ below.

The parameter ϕ represents the strength of contract enforcement in the environment. One can think about ϕ as an enforcement cost, a deadweight loss resulting from the default process, or the amount of resources with which the borrower can abscond. The exact interpretation is unimportant for the moment, so I will refer to it as an enforcement cost. Throughout this paper I model limited enforcement in a reduced-form fashion. This is a common approach in the literature on financial frictions and misallocation, but one can conceive of a structural interpretation based on a model like Kehoe and Levine (1993). In the quantitative version of the model in section 5 I adopt the specification of D’Erasmus and Boedo (2012), using two enforcement-related parameters that map directly to measures reported in *Doing Business*.

Before fully specifying the environment I want to use the general form of the break-even condition above to illustrate how imperfect information about default risk interacts with the cost of enforcing contracts to affect the interest rate the lender will charge. Consider an extreme example in which the risk-free rate r^* is zero and there is no enforcement cost ($\phi = 0$). In this case, the break-even condition reduces to

$$\ell = (1 - \mathbb{E}[p(\ell, r)])(1 + r)\ell + \mathbb{E}[p(\ell, r)]\ell \quad (3)$$

Clearly, the only interest rate on the loan that satisfies the break-even condition is zero – the lender always gets back ℓ regardless of whether the borrower repays or defaults. As a consequence, any uncertainty about the default probability is irrelevant. When the enforcement cost is positive, however, the lender’s payoff is risky and the interest rate r must compensate for this risk. As a consequence, uncertainty about the borrower’s default risk affects the interest rate. When the enforcement cost is small, there is very little risk in the lender’s payoff from making the loan, so the interest rate will be close to the risk-free rate regardless

²The notion that lenders might be uncertain about default risk is not entirely new to the equilibrium default literature. For example, Pouzo and Presno (2012) study how investors’ concerns about model misspecification affect sovereign bond spreads. To my knowledge, my paper is the first to study how these issues affect resource allocation.

of how uncertain the lender is about the thd default probability. But when the enforcement cost is high, the lender's payoff becomes more risky and the interest rate becomes more sensitive to uncertainty about the borrower's default risk.

I now move to a complete description of the market structure. There is a large number of borrowers that differ in their types $b \in B$ which are distributed according to a distribution $G(b)$. In this section I abstract from any modeling of borrowers' preferences or maximization problems. In the full model in the next section these types will be related to firm's productivities, but for the moment I assume that a borrower's type simply determines her default probability, which is a function $p_b(\ell, r)$ that depends on the loan size ℓ and the interest rate r . I assume that lenders know the distribution G and the functions p_b , but that they do not observe borrowers' types b directly. Instead, they observe noisy signals $c = b + \eta$, where the noise term η is drawn from a distribution $H(\eta)$. All lenders observe the same signal about a particular borrower. I assume that lenders know the distribution H as well. The distributions G and H induce a conditional distribution $G(b|c)$ that gives the probability of a borrower's type being b given that lenders observe the signal c . I make two important assumptions. First, borrowers cannot inform lenders, truthfully or otherwise, about their types. Second, a borrower's choice of loan size does not reveal any information about her type – lenders know the mapping $b \mapsto p_b$ but nothing about borrowers' preferences, etc.

Before moving on, I wish to make several comments about these assumptions, since the information framework in my model differs from others that are often used in the literature on finance and misallocation in several respects. One common interpretation of imperfect information is an environment in which lenders have no information about individual borrowers' types, but lenders have perfect information about the mapping between borrowers' types and their preferences, production technologies, etc. The primary concern from a lender's perspective in this kind of environment is that borrowers may lie about their types to obtain more favorable contact terms, but lenders can typically design incentive-compatible contracts to elicit truth-telling. Examples of this approach to imperfect information in the quantitative literature on finance and misallocation are studies like Greenwood, Sanchez, and Wang (2010), Erosa and Cabrillana (2008) and Neira (2012). This kind of framework often allows for a variety of contingencies to be built into contracts, but in my model I allow for only one contingency: default. More importantly, however, I take a different stance on the information available to lenders. On the one hand, I allow for lenders to have some information, albeit imperfect, about borrowers' types in the form of noisy signals. On the other hand, I assume that lenders do not know everything about the mapping from a borrower's type to characteristics – preferences, production technology, etc. Instead, I assume that lenders know only that a borrower's type is associated with a particular default probability. Given this assumption, there is no way for lenders to make use of the revelation principle or infer additional information from borrowers' choices, so they simply use

the information available to them – noisy signals – to estimate each borrower’s default risk and use those estimates to set borrower-specific contract terms. This market structure yields a tractable model in which the effect of the interaction between imperfect information about default risk and limited contract enforcement on TFP is consistent with the empirical evidence outlined above.³ One way to modify my approach to allow lenders to make some inferences from borrowers’ decisions while maintaining the incomplete markets structure with equilibrium default would be to adopt the framework in Chatterjee, Corbae, and Ríos-Rull (2008). This, however, would greatly reduce the model’s tractability.

My approach is intended to capture the notion that lenders use data, like that available in public credit bureaus or private credit registries, to determine that borrowers with certain traits are more likely to default on loans of certain sizes. In countries that lack these kinds of institutions, this data is scarcer and likely to be less reliable, leading to less accurate estimates of borrowers’ default risk. This idea applies to information about prospective borrowers themselves, but also to information about other borrowers in the past with which lenders can compare prospective borrowers. The noisy signals lenders receive about borrowers are intended to serve as a reduced-form representation of this idea. One might envision, however, that these signals could represent a technology that lenders use to transform information provided by borrowers into an estimate of default risk. As an example of the kind of uncertainty η is intended to capture, consider the following scenario motivated by the depth of credit measure used in the regression in section 2. Consider a firm that wants to take out a loan to expand its operations. The firm brings all the information it can provide (e.g. balance sheet, cash flow statement, etc.) to its lender, who is able to verify this information by visiting the firm’s offices, checking its bank statements, etc. The lender uses this information to assess the likelihood that the firm will default on a given loan. In a country with a credit bureau that has extensive coverage of borrowers and their histories, the lender will be able to use this database to determine the repayment performance of similar borrowers in the past to construct an accurate estimate of the firm’s repayment probability. In a country with no credit bureau or one with narrower coverage, it will be more difficult for the lender to conduct this kind of analysis, leading to a less accurate prediction about the firm’s likelihood of repayment. The variance of the noise term η (or more precisely its inverse) represents the quality of the credit information environment. In an economy with a high variance, lenders are less able to accurately assess firms’ default probabilities. As I will show, my approach yields an analytically and numerically tractable framework that clearly illustrates how limited contract enforcement and imperfect information together effect create misallocation that reduces TFP.

³The effect of the interaction between limited contract enforcement and imperfect information on misallocation and TFP has not been a focus in the literature. However, Neira (2012) analyzes a model with limited enforcement and a conventional asymmetric information problem using optimal contracting techniques. He finds that the information problem has a smaller effect on TFP when contract enforcement is weak, which is inconsistent with the data.

Given the information structure in this model, lenders use their signals c together with their knowledge of the distributions G and H and the map $b \mapsto p_b$ to construct an estimate of a borrower's default probability. The probability that a borrower about which lenders observe signal c defaults on a loan of size ℓ at interest rate r is

$$\mathbb{E}[p_b(\ell, r)|c] = \int_B p_b(\ell, r) dG(b|c) \quad (4)$$

Lenders then use this estimate to set the interest rate r on each loan size ℓ they are willing to make to the borrower. In particular, lenders use this estimate of the borrower's default probability to construct two objects: a loan set $\mathbf{L}(c)$ and an interest rate schedule $\mathbf{r}(\cdot, c) : \mathbf{L}(c) \rightarrow \mathbb{R}_+$. I use bold font here to distinguish the interest rate schedule, a function, from a scalar interest rate r . The loan set gives the loan sizes lenders are willing to make and the interest rate schedule gives the interest rate they charge. Given the assumptions above that borrowers cannot communicate their types (again, truthfully or otherwise) and that a borrower's choice of loan size contains no additional information for lenders, the interest rate $\mathbf{r}(\ell, c)$ on a given loan $\ell \in \mathbf{L}(c)$ is set to that lenders expect to break even on the loan conditional on the borrower taking it. Hence for all $\ell \in \mathbf{L}(c)$, the interest rate $\mathbf{r}(\ell, c)$ satisfies

$$(1 + r^*)\ell = (1 - \mathbb{E}[p_b(\ell, \mathbf{r}(\ell, c))|c]) (1 + r)\ell + \mathbb{E}[p_b(\ell, \mathbf{r}(\ell, c))|c] (1 - \phi)\ell \quad (5)$$

The loan set $\mathbf{L}(c)$ is simply the set of all loan sizes ℓ for which a solution to this equation exists. In general, there may be more than one solution to this equation. In the rest of the paper, I assume that $\mathbf{r}(\ell, c)$ is equal to the smallest solution. This is an innocuous assumption, however. One can imagine how it might derive from competitive pressure, but we could equivalently treat $\mathbf{r}(\cdot, c)$ as a correspondence that can take multiple values and allow borrowers to choose from among them; in the production model below borrowers will always choose the smallest interest rate associated with a particular loan size.

In the next section I incorporate this framework into a production model with heterogeneous firms to study the effects this information friction, together with limited contract enforcement, on resource allocation and total factor productivity. In the following section I extend the model to a dynamic, quantitative setting in which firms can accumulate capital over time.

4 Stylized model

In this section I build on the framework described in the previous section, fleshing out the production side of the economy to illustrate how imperfect information about borrowers causes misallocation of resources and how limited enforcement of contracts exacerbates this effect. I keep things simple to aid in analytical

characterization, studying a two-period model that essentially static. There is a unit measure of firms and, as before, a large number of risk-neutral, perfectly competitive lenders. The economy is endowed with a fixed capital stock \bar{K} . The risk-free rate r^* is exogenous.⁴

In the second period, firms produce output using capital k in a decreasing returns technology

$$y = e^a k^\alpha, \quad \alpha \in (0, 1) \quad (6)$$

where a is a firm's (log) productivity. Firms are heterogeneous in k and a . Firms' productivities a are distributed according to a distribution $F(a)$, while their capital stocks are determined by their investment decisions in the first period. For analytical convenience I assume that F is normal with mean zero and variance σ_a^2 . Let Λ denote the joint distribution of a and k in the second period. Aggregate output is given by

$$Y = \int e^a k^\alpha d\Lambda(a, k) \quad (7)$$

Λ , an endogenous object, is the direct driver of the economy's aggregate productivity. Since firms have decreasing returns to scale it's optimal⁵ to allocate some capital to each firm, but more productive firms (those with higher a) ought to receive more capital. Frictions that reduce the correlation between a and k will reduce aggregate productivity. As we will see, the information friction I study in this paper does precisely this.

In the first period, firms choose how much capital to purchase. The price of capital in the first period is p . Firms enter the first period with no resources so they must borrow from lenders to finance this investment. Each firm is matched with a lender. A firm that chooses to purchase capital k must take out a loan $\ell = pk$ at an interest rate r which will be determined by its lender in a process I will describe shortly. Firms do not learn their productivities a until period two. In the first period, firms receive signals $b = a + \epsilon$ about their actual productivities, where ϵ is a normally distributed noise term with variance σ_ϵ^2 . This uncertainty from the firm's point of view makes its technology risky, creating the possibility that its second-period revenues $y = e^a k^\alpha$ will be insufficient to cover the amount $\ell(1 + r)$ it is supposed to repay the lender. Firms have limited liability and can default on their loans and shut down if this situation arises. If a firm defaults, its lender recovers the loaned funds after paying a cost $\phi \in [0, 1]$. This parameter governs the strength of contract enforcement in the economy; higher values of ϕ mean enforcing contracts is more costly for

⁴For the moment, one can think about this as being a small open economy or a closed economy in which the risk-free rate is determined in equilibrium by consumers' discount factor. The distinction is unimportant at this stage, although I adopt the latter assumption in the quantitative model of the next section.

⁵I use the term *optimal* in this context to refer to the allocation that a hypothetical planner who could costlessly allocate capital in period two would choose. Due to the timing and information structure in the model the competitive equilibrium will deviate from the solution to this planner's problem, even in the absence of noise and enforcement frictions. Nevertheless, this planner's problem provides useful insight into the determinants of aggregate TFP in the model.

lenders. The interest rate r must compensate the lender for this default risk. The firm's signal b is the analogue of its "type" in the sense of the previous section.

Lenders do not directly observe the firms' signals b . Instead, lenders receive signals $c = b + \eta$, where η is another normally distributed noise term with variance σ_η^2 . As in the previous section, I assume that firms cannot communicate, truthfully or otherwise, any information about their signals to lenders and that firms' investment decisions do not contain any information for lenders, either. In other words, lenders have no knowledge of firms' technologies or how and why firms make their investment decisions; they know only that each type b is associated with a default probability function $p_b(k, r)$, which I define in terms of capital rather than loan size, given the one-to-one correspondence between the two. Given this information friction, lenders can offer loan contracts conditional only on c and the amount of capital the firm purchases. Each firm's lender offers a set of contracts composed of two objects, an investment set $\mathbf{K}(c)$ and an interest rate schedule $\mathbf{r}(\cdot, c) : \mathbf{K}(c) \rightarrow \mathbb{R}_+$. The investment set describes the range of capital purchases the lender is willing to finance. The interest rate schedule describes the interest rate the lender charges on each possible investment. In other words, for each $k \in \mathbf{K}(c)$, the lender charges an interest rate of $\mathbf{r}(k, c)$. Again, since there's a one-to-one correspondence between loan size and capital in this setting, I define these objects in terms of capital rather than loan sizes (hence $\mathbf{K}(c)$ rather than $\mathbf{L}(c)$). Due to competitive pressure the lender must break even in expectation on each contract $(k, \mathbf{r}(k, c))$.

4.1 Firm's problem

Firms take the price of capital p , the set of investment opportunities $\mathbf{K}(c)$ and the interest rate schedule $\mathbf{r}(\cdot, c) : \mathbf{K}(c) \rightarrow \mathbb{R}_+$ as given. The information structure in the model implies that, from the firm's perspective, the lender's signal c does not add any additional information about the firm's actual productivity a .

The most intuitive way to describe the firm's problem is to work backwards, starting with the default decision the firm makes after learning its true productivity. Conditional on having chosen capital k and borrowed the amount $\ell = pk$ at the (scalar) interest rate r , at this stage the firm simply chooses whether to produce and pay back its loan or to default. In truth, this is not really a choice at all; if the firm can generate sufficient revenue to repay its loan it will do so, but if it cannot it must default. If the firm defaults it forgoes production and exits with a payoff of zero. Define $v_2(a, k, r)$ as the value of a firm with productivity a , capital k and interest rate r in the second period:

$$v_2(a, k, r) = \max\{e^a k^\alpha - (1 + r)pk, 0\} \quad (8)$$

The default “decision” is characterized by a cutoff value \underline{a} such that

$$e^{\underline{a}}k^\alpha - (1+r)pk = 0 \quad (9)$$

For all values of a below \underline{a} the firm will default. As we will see shortly, it will be useful to treat \underline{a} as a function of capital k and a scalar interest rate r :

$$\underline{a}(k, r) = \log \left(\frac{(1+r)pk}{k^\alpha} \right) \quad (10)$$

Moving backwards in time, consider now the problem of a firm with signal b in the first period. The firm’s objective at this stage is to choose capital k to maximize its expected second-period value:

$$v_1(b, c) = \max_{k \in \mathbf{K}(c)} \mathbb{E} [v_2(a, k, \mathbf{r}(k, c)) | b] \quad (11)$$

Let $k(b, c)$ denote the firm’s optimal investment policy. To characterize the solution to this problem, it is convenient to re-write it as

$$v_1(b, c) = \max_{k \in \mathbf{K}(c)} \left\{ \int_{\underline{a}(k, \mathbf{r}(k, c))}^{\infty} [e^a k^\alpha - (1 + \mathbf{r}(k, c))pk] dF(a|b) \right\}$$

where $F(a|b)$ is the density of a conditional on the signal b . Applying the usual formula for Bayesian updating with normal distributions, we see that $F(a|b)$ is normal with mean $\mu_{a|b}$ and variance $\sigma_{a|b}^2$, given by

$$\mu_{a|b} = \left(\frac{\sigma_\varepsilon^{-2}}{\sigma_a^{-2} + \sigma_\varepsilon^{-2}} \right) b, \quad \sigma_{a|b}^2 = \frac{1}{\sigma_a^{-2} + \sigma_\varepsilon^{-2}} \quad (12)$$

The first-order condition of this problem is

$$\alpha k^{\alpha-1} \int_{\underline{a}(k, \mathbf{r}(k, c))}^{\infty} e^a dF(a|b) = [(1 + \mathbf{r}(k, c))p + \mathbf{r}_k(k, c)pk] [1 - F(\underline{a}(k, \mathbf{r}(k, c)) | b)] \quad (13)$$

This condition is this model’s analogue of standard marginal product pricing. The left-hand side is the expected marginal product of capital, while the right-hand side is the expected marginal cost. Note, however, the presence of the derivative of the interest rate schedule (which exists everywhere on the interior of $\mathbf{K}(c)$ as shown in appendix A) on the right-hand side. In choosing its optimal level of investment, the firm takes into account the fact that an additional unit of capital changes the interest rate its lender charges.

4.2 Lender's problem

Just as in the previous section, the interest rate function $\mathbf{r}(k, c)$ must satisfy the break-even condition

$$(1 + r^*)\ell = (1 + \mathbf{r}(k, c))\ell(1 - \mathbb{E}[p_b(k, \mathbf{r}(k, c))|c]) + (1 - \phi)\ell\mathbb{E}[p_b(k, \mathbf{r}(k, c))|c], \forall k \in \mathbf{K}(c) \quad (14)$$

where $\ell = pk$ as before and $\mathbb{E}[p_b(k, r)|c]$ is the probability that firm the firm defaults conditional on the lenders' signal c and amount of investment k . The left-hand side is the return on investing ℓ at the risk-free rate r^* . The right-hand side is the return on loaning ℓ the firm. The terms $(1 + \mathbf{r}(k, c))\ell$ and $(1 - \phi)\ell$ are what the lender gets if the firm repays and defaults respectively, and these two possibilities are weighted by the probabilities they they occur. The set of investment projects $\mathbf{K}(c)$ offered to the firm is simply the set of capital values for which there exists a solution to equation (14).

Conditional on having received the signal b , the probability that the firm will default if it buys capital k at interest rate r is

$$p_b(k, r) = F(\underline{a}(k, r)|b) \quad (15)$$

Lenders only know the shape of this function, not the objects (the firm's technology, the distribution of a , etc.) that give rise to it. Again, just like in the previous section, We can obtain the lender's estimate of the default probability, $\mathbb{E}[p_b(k, r)|c]$, as follows:

$$\mathbb{E}[p_b(k, r)|c] = \int_{-\infty}^{\infty} p_{c-\eta}(k, r) d\Phi(\eta/\sigma_\eta) \quad (16)$$

where Φ is the standard normal distribution.

4.3 Aggregation and equilibrium

Aggregation and equilibrium in this economy are straightforward. There is one caveat: I assume that capital recovered from defaulting firms re-enters the economy so that the entire endowment of aggregate capital \bar{K} is used in production. One can imagine that lenders re-sell capital recovered from defaulting firms to a second round of entering firms and so on. This is not a critical assumption – none of my results with this two-period model, qualitative or quantitative, depend on it. Under the alternative assumption that capital purchased by defaulting firms does not get used to produce, there is another margin for variation in aggregate output in addition to misallocation of utilized capital. I choose to avoid this. This assumption plays no role in the dynamic model in the next section.

For convenience, let $P(a, b, c)$ denote the joint distribution over firms' true productivities a , their own

signals b , and lenders' signals c . Abusing notation slightly, let $\underline{a}(b, c)$ denote the cutoff productivity function evaluated at firms' equilibrium investment choices:

$$\underline{a}(b, c) = \underline{a}(k(b, c), \mathbf{r}(k, c)) \quad (17)$$

The total mass of firms that produce is

$$M = \int \mathbb{1}_{\{a \geq \underline{a}(b, c)\}} dP(a, b, c) \quad (18)$$

Aggregate capital employed in production is

$$K = \int k(b, c) \mathbb{1}_{\{a \geq \underline{a}(b, c)\}} dP(a, b, c) \quad (19)$$

Aggregate output is

$$Y = \int e^a k(b, c)^\alpha \mathbb{1}_{\{a \geq \underline{a}(b, c)\}} dP(a, b, c) \quad (20)$$

Total factor productivity is

$$A = \frac{Y}{K^\alpha M^{1-\alpha}} \quad (21)$$

The assumption of decreasing returns to scale in production implies that aggregate output Y is a function of the mass M of firms (which varies depending on how many firms' true productivities are below their cutoffs) in addition to aggregate capital stock K , hence M enters as a "factor" into the calculation of TFP (see proposition 9 in appendix A). Midrigan and Xu (2010) point out that this is similar to the love-for-variety effect in models with monopolistic competition and Dixit-Stiglitz preferences. In practice, only a small percentage of firms default so there is little variation in M .

An equilibrium in this environment is a simple object. The only market that need clear is the capital market, so the price of capital is the only endogenous price (in addition to the interest rate schedules). The formal definition of equilibrium follows.

Definition 1. *A competitive equilibrium is*

- *Investment sets and interest rate schedules $\mathbf{K}(c), \mathbf{r}(\cdot, c)$*
- *Investment policies $k(b, c)$*
- *Capital price p*

such that investment sets and interest rates satisfy lenders' break-even condition (14), firms' investment policies solve

their problem (12), and the capital market clears: $K = \bar{K}$.

4.4 Characterization

The model described above is simple, but the interest rate schedules on which everything else hinges have no analytical solution. In lieu of presenting explicit expressions for the key model objects, in this section I provide a combination of mathematical results and illustrations (obtained by numerical approximation) to shed some light on the interest rate schedules' key properties and the consequences of these properties for misallocation of capital across firms.

4.4.1 Interest rate schedules

It is helpful to first simplify the lender's break-even condition (14) by solving for the distribution of the firm's true productivity a conditional on the lender's signal c :

Proposition 1. *Conditional on c , a is normally distributed with mean $\mu_{a|c}$ and variance $\sigma_{a|c}^2$, given by*

$$\mu_{a|c} = \left(\frac{\sigma_\varepsilon^{-2}}{\sigma_a^{-2} + \sigma_\varepsilon^{-2}} \right) \left(\frac{\sigma_\eta^{-2}}{\sigma_b^{-2} + \sigma_\eta^{-2}} \right) c, \quad \sigma_{a|c}^2 = \sigma_{a|b}^2 + \left(\frac{\sigma_\varepsilon^{-2}}{\sigma_a^{-2} + \sigma_\varepsilon^{-2}} \right)^2 \left(\frac{1}{\sigma_b^{-2} + \sigma_\eta^{-2}} \right) \quad (22)$$

where $\sigma_b^2 = \sigma_a^2 + \sigma_\varepsilon^2$.

Let $F(a|c)$ denote the CDF associated with this distribution. Notice that as σ_η approaches zero, $F(a|c)$ converges to $F(a|b)$ as the lenders get closer to perfectly accurate knowledge about b . Conversely, as σ_η approaches ∞ , $\mu_{a|c} \rightarrow 0$ since that the lender's signal c contains no information at all about the value of b . I stress here that since lenders don't the firms' technology, the distribution of a , etc., this result has no bearing on lenders' thinking in setting loan contracts. I can still use it, however, to characterize equilibrium objects.

We can now write the break-even condition as

$$(1 + r^*)\ell = (1 + \mathbf{r}(k, c))\ell [1 - F(\underline{a}(k, \mathbf{r}(k, c))|c)] + (1 - \phi)\ell F(\underline{a}(k, \mathbf{r}(k, c))|c) \quad (23)$$

This is the key equation that determines investment sets $\mathbf{K}(c)$ and interest rate schedules $\mathbf{r}(k, c)$. To characterize the solution it is helpful to define the following auxiliary function

$$\omega(k, r, c) = (1 + r) [1 - F(\underline{a}(k, r)|c)] + (1 - \phi)F(\underline{a}(k, r)|c) - (1 + r^*) \quad (24)$$

This function gives the expected return in per-unit terms to a lender with signal c if the firm buys capital k

at scalar interest rate r . If $\omega(k, r, c)$ is positive, the lender expects to profit from the loan, while if $\omega(k, r, c)$ is negative lenders expect to take a loss. The break-even condition that pins down the values of the interest rate schedule $\mathbf{r}(k, c)$ can be expressed as $\omega(k, \mathbf{r}(k, c), c) = 0$. In general, if we hold k fixed $\omega(k, \cdot, c)$ has at most two roots. Competition among lenders implies that $\mathbf{r}(k, c)$ is the smaller of the two. For large enough k , however, $\omega(k, r, c)$ is always negative. For such k there is no interest rate at which lenders can expect to at least break even. As a consequence, lenders will not make loans for such k at all.

Let $\bar{\omega}(k, c) = \max\{\omega(k, r, c) | r \geq r^*\}$. This function gives the maximum expected payoff the lender could achieve on a loan for investment k to the firm, leaving aside any issues of competition. It is straightforward to show that there exists a unique $\bar{k}(c)$ that satisfies $\bar{\omega}(\bar{k}(c), c) = 0$. At $\bar{k}(c)$ the maximum expected payoff the lender could achieve at any interest rate is exactly zero. For $k > \bar{k}(c)$ the lender cannot at least break even in expectation at any interest rate. These upper bounds $\bar{k}(c)$ are all that we need to define the investment sets $\mathbf{K}(c)$, which are simply closed intervals $[0, \bar{k}(c)]$. It should come as no surprise to the reader that $\bar{k}(c)$ is increasing in c . In other words, firms that are viewed more optimistically by lenders can borrow more. The following propositions formally establish these results.

Proposition 2. *For each c , the following is true about $\bar{\omega}(k, c)$:*

- (i) *It is continuous.*
- (ii) *It is strictly decreasing.*
- (iii) $\lim_{k \rightarrow 0} \bar{\omega}(k, c) = \infty$.
- (iv) $\lim_{k \rightarrow \infty} \bar{\omega}(k, c) = \phi - r^*$.

Proposition 3. *For each c , there exists a unique $\bar{k}(c)$ such that $\bar{\omega}(\bar{k}(c), c) = 0$. Moreover, $\bar{k}(c)$ is increasing in c .*

The implication of this result is that firms that are viewed too pessimistically by lenders ($c < b$) cannot invest as much as they would be able to in the absence of the information friction. Conversely, firms that are viewed too optimistically ($c > b$) can invest more. This causes misallocation of capital across firms.

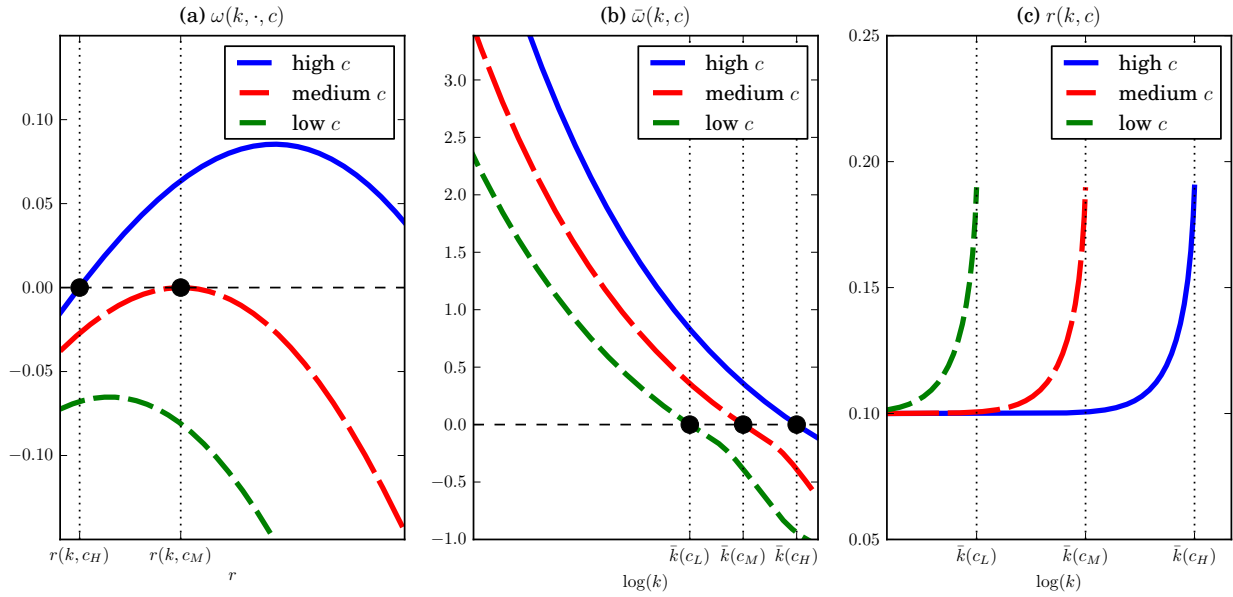
Having characterized the investment sets I move on to the interest rate schedules themselves. As the firm borrows more its default probability increases so lenders must charge higher interest rates to break even in expectation. This means that the interest rate schedule for each firm is strictly increasing. Holding fixed the amount of investment k , firms that lenders view more optimistically (higher c) are less likely to default, so $\mathbf{r}(k, c)$ is decreasing in c .

Proposition 4. *Holding fixed c , the interest rate schedule $\mathbf{r}(k, c)$ is increasing in k . Holding fixed k , $\mathbf{r}(k, c)$ is decreasing in c .*

The implication of this result is that firms that are viewed too pessimistically by lenders ($c < b$) pay interest rates that are too high relative to their default probabilities, while firms that are viewed too optimistically ($c > b$) pay interest rates that are too low. This provides an additional channel for misallocation of capital.

Figure 2 below provides a complete graphical depiction of the above results. In panel (a), I plot the arbitrage function $\omega(k, r, c)$ over a range of interest rates for three different levels of investment k . The blue line plots $\omega(k, r, c)$ for $k < \bar{k}(c)$. We can see that the arbitrage function indeed crosses zero twice. The value of the interest rate schedule $r(k, c)$ is indicated on the graph. The red line plots $\omega(\bar{k}(c), r, c)$, showing that it exactly touches zero at its peak. The green line plots $\omega(k, r, c)$ for $k > \bar{k}(c)$, and we see that it is indeed always negative. In panel (b) I plot $\bar{\omega}(k, c)$ for three different values of the lender signal c . The blue line represents a high signal, the red line a medium signal, and the green line a low signal. The upper bounds $\bar{k}(c)$ are indicated on the graph, and we can see that they increase as the lender signal increases. Finally, in panel (c) I plot the interest rate schedules themselves for the same three lender signals as in panel (b).

Figure 2: Determination of investment sets and interest rate schedules



One of this paper's main points is that the cost of enforcing contracts, represented by ϕ , exacerbates the effects of the information friction captured by σ_η . This result is driven entirely by the fact that increasing the enforcement cost makes loan terms more sensitive to default risk. The information friction manifests itself by distorting lenders' estimates of firms' default risk, so as loan terms become more sensitive to default risk, they also become more sensitive to the effects of the information friction.

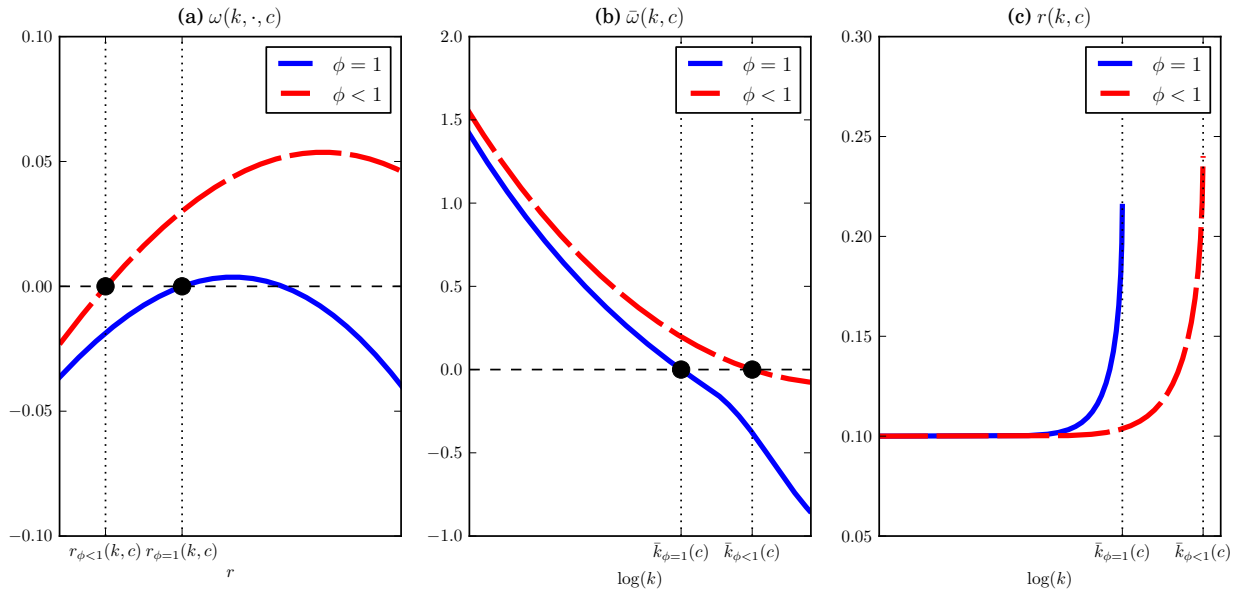
As the cost of enforcing contracts falls (or conversely as the recovery rate $1 - \phi$ rises), holding k and r fixed the payoff function $\omega(k, r, c)$ increases. This causes $\mathbf{r}(k, c)$, the first value of r at which $\omega(k, \cdot, c)$ crosses zero, to fall and causes the upper bounds $\bar{k}(c)$ to rise. In short, decreasing ϕ makes loan terms more sensitive to default risk. While I cannot formally prove that these effects hold in general equilibrium (varying ϕ affects the equilibrium price of capital, which in turn affects interest rate schedules), my quantitative results indicate that they do. As a complement, I provide below two partial equilibrium results that formalize these effects assuming that the price p of capital remains fixed:

Proposition 5. *Holding fixed the price p of capital,*

- (i) $\bar{k}(c)$ is decreasing in ϕ for each c .
- (ii) $\mathbf{r}(k, c)$ is increasing in ϕ for each c and each k .

Figure 3 below illustrates this result by plotting the same functions as in figure 2, this time comparing versions of the model with $\phi = 1$ (high enforcement costs) and with $\phi < 1$ (low enforcement costs). In panel (a), we see that when $\phi < 1$ the arbitrage function $\omega(k, r, c)$ crosses zero earlier, lowering the value of $\mathbf{r}(k, c)$. In panel (b) we see that $\bar{\omega}(k, c)$ crosses zero later, increasing the upper bounds $\bar{k}(c)$. And in panel (c) we see the results: firms in the economy with $\phi < 1$ can borrow more and they pay lower interest rates.

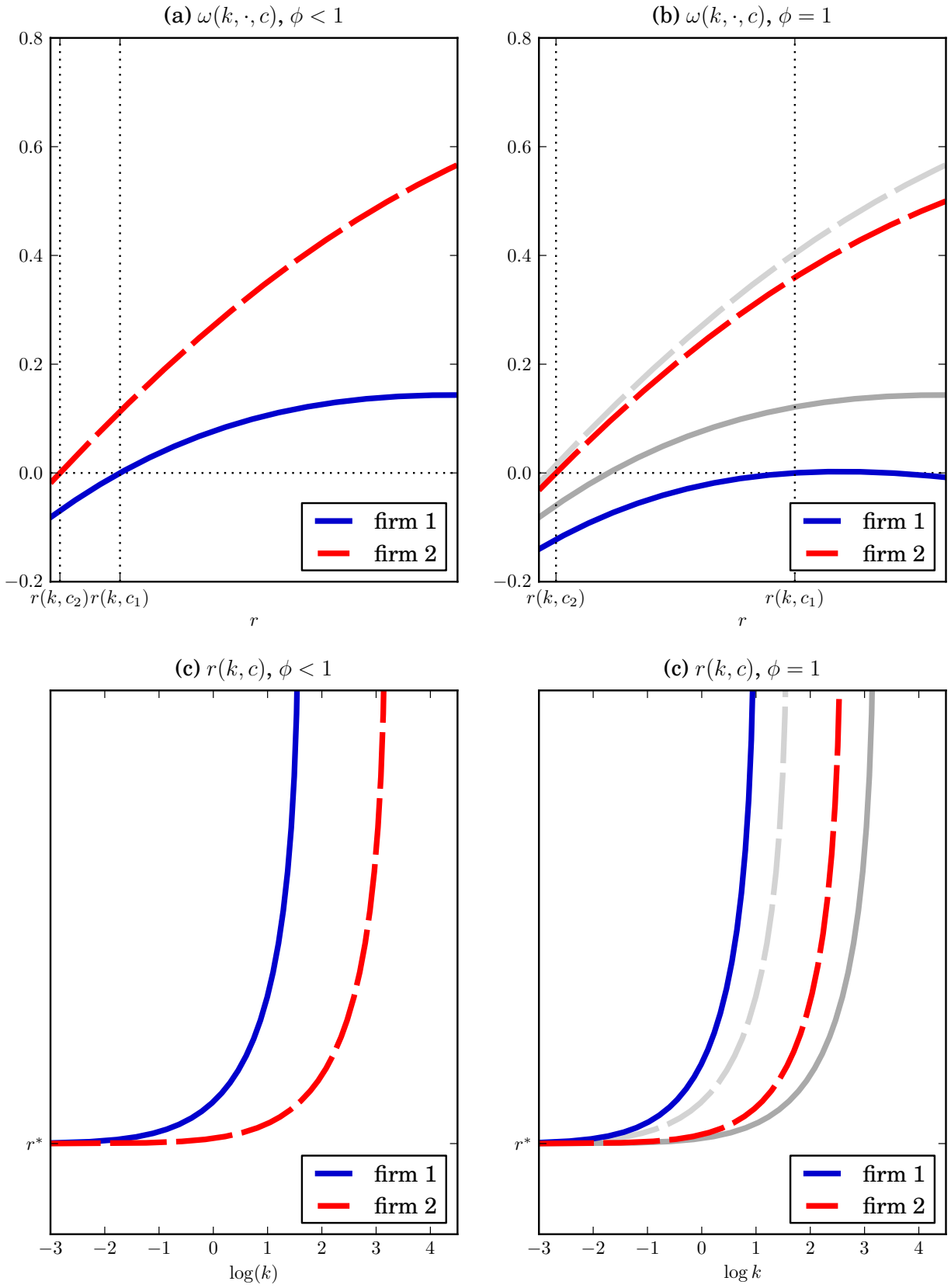
Figure 3: Effects of enforcement cost on loan terms



As ϕ increases and loan terms become more sensitive to default risk, the information friction has a larger impact on those terms as well. To see this, consider two firms with the same signal b about their own

productivities, but who are viewed differently by lenders. Let $c_1 < c_2$ be the lenders' signals about these two firms, so that firm 1 is viewed as more risky than firm 2. In panels a and b of figure 4 I plot lenders' arbitrage functions for these two firms and their interest rate schedules' values for a fixed investment size, just as in the first panel of the two previous figures. Panel a plots these functions for $\phi < 1$ (low enforcement cost), while panel b plots them for $\phi = 1$ (high enforcement cost). I plot panel a's lines in grey in panel b to make it clear how they are changing. Because firm 1 is viewed as a high risk, increasing the enforcement cost has a large impact on the interest rate it pays on this loan size due to the previous result. Firm 2, on the other hand, is viewed as a low risk, so increasing the enforcement cost doesn't have much impact on its interest rate. Panels c and d plot the interest rate schedules for the same two firms in the same two scenarios. When the enforcement cost is low, both firms pay similar, relatively low, interest rates until the investment size becomes large. When the enforcement cost is high, firm 1's interest rate starts to rise at for smaller investments. In short, the two firms face larger differences in the marginal cost of investing for smaller investments as the enforcement cost rises.

Figure 4: Effects of enforcement cost on loan terms' sensitivity to information friction



4.4.2 Equilibrium capital allocations

When I introduced the model above, I emphasized that the joint distribution of capital and productivity is the key determinant of aggregate productivity. I cannot derive analytical expressions for firms' equilibrium allocations of capital, but I can provide a partial characterization that sheds some light on the mechanisms driving misallocation by analyzing the dispersion of the marginal product of capital. In the empirical literature on misallocation, studies often focus on dispersion in the marginal products of productive factors across establishments or firms because these measures indicate how much an economy deviates from a frictionless benchmark model in which marginal products are equalized across establishments/firms. For example, Hsieh and Klenow (2009) find that the standard deviation of revenue productivity⁶ is significantly higher in China and India than in the U.S. As we will see, the information friction in my model has a similar effect.

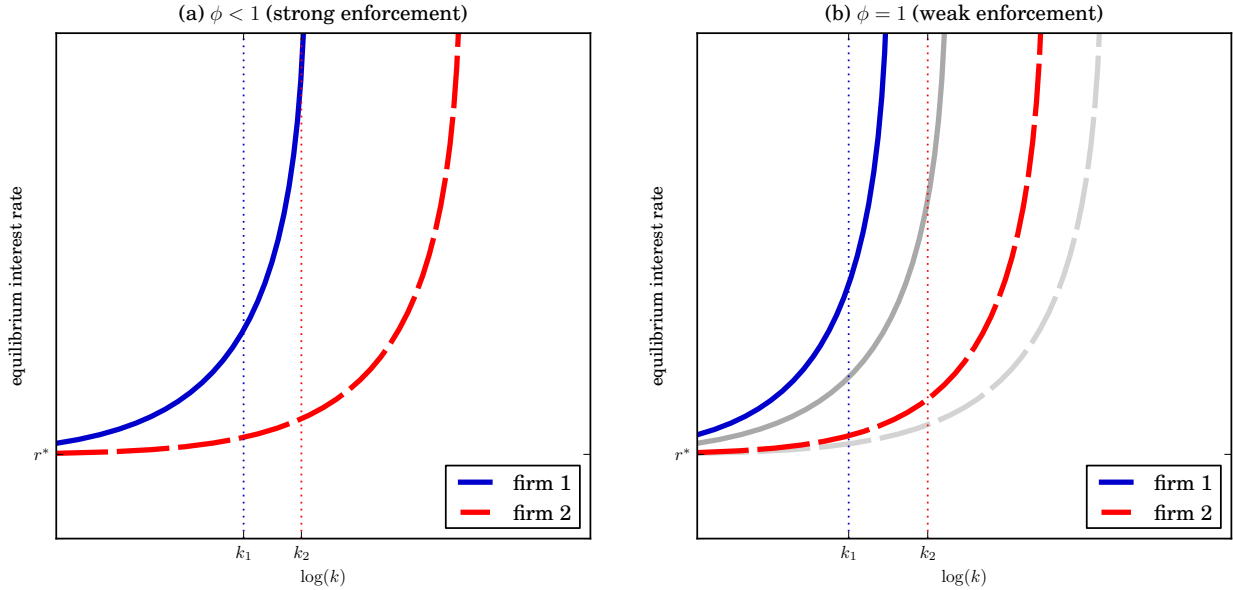
One way to illustrate how the model generates misallocation is to compare the equilibrium capital allocations of two firms that receive the same signal b (and hence have the same expected productivity) but about whom lenders have different default probability estimates. In the perfect-information version of the model, two such firms would receive the same allocations of capital, while in the imperfect-information version they receive different allocations: the firm that lenders view as less likely to default (i.e. the firm for which the lenders' signal c is higher) receives a larger allocation of capital and thus produces at a larger scale. Moreover, the difference on allocations between the two firms is larger when the enforcement cost is higher – weak contract enforcement increases the variation in capital allocations received by firms with the same expected productivity. The following results formalize this point:

Proposition 6. *Fix b and $c_1 < c_2$. Let k_1 and k_2 denote $k(b, c_1)$ and $k(b, c_2)$. Then, holding p fixed, $\log k_2 - \log k_1$ is increasing in ϕ .*

Figure 5 illustrates the logic behind this result graphically. In the figure, I use “firm 1” to denote the firm with the lower lender signal and “firm 2” to denote the firm with the higher lender signal. When the contract enforcement cost is low (panel a), the interest rate schedules these two firms receive are not very sensitive to the effects of the information friction, and as a consequence their interest rate schedules take similar values over a large range of investment sizes. This leads the two firms to make similar investments. When the enforcement cost is high (panel b), the interest rate schedules become more sensitive to the effects of the information friction, so the difference between the two schedules' values becomes large even for small investment sizes. This leads the two firms to make less similar investments. In panel b, I plot the strong-enforcement interest rate schedules in grey to make it clear where they're moving from.

⁶A geometric average of the marginal products of capital and labor; often called TFPR.

Figure 5: Illustration of misallocation between firms with the same b



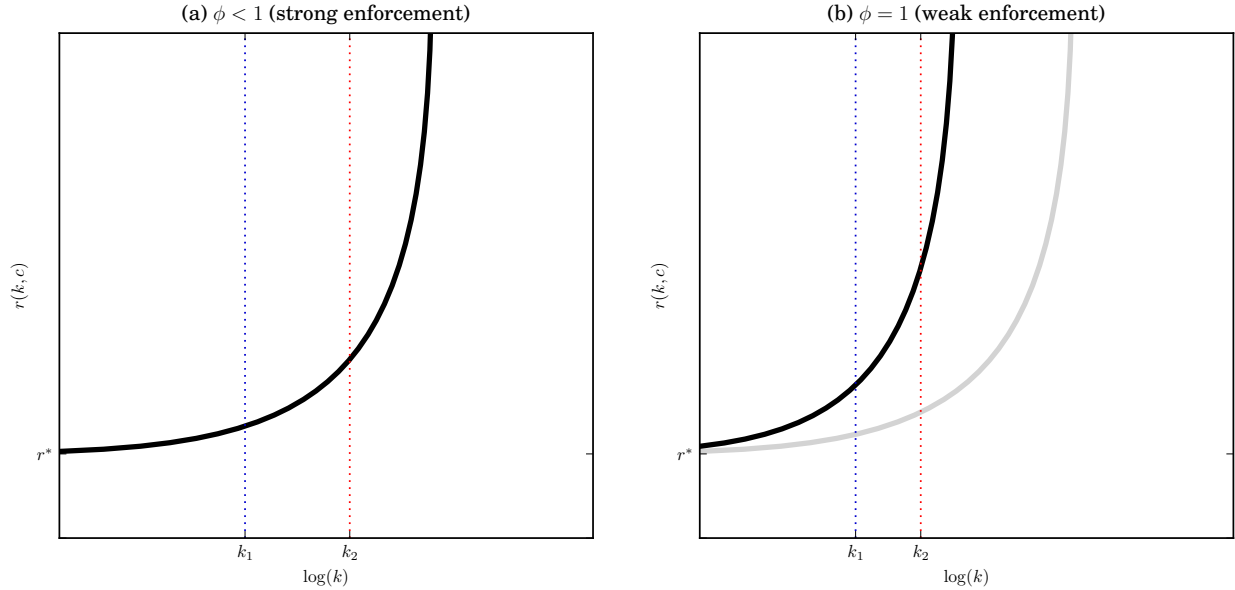
Another way to illustrate the misallocation mechanism is to compare the equilibrium capital allocations of two firms that have different values of their own signals b , but about whom lenders receive the same signal c . In a perfect-information world, these two firms would receive capital allocations that reflect their ex-ante differences in expected productivity. In the imperfect-information version, this difference is dampened by the fact that lenders view the two firms as equal risks.

Proposition 7. Fix $b_1 < b_2$ and c . Let k_1 and k_2 denote $k(b_1, c)$ and $k(b_2, c)$. Then, holding p fixed, holding p fixed, $\log k_2 - \log k_1$ is decreasing in ϕ .

Figure 6 illustrates the logic behind this result graphically, using the same notation as before. When the contract enforcement cost is low (panel a), the interest rate schedule (now singular) these two firms receive is not very sensitive to the effects of the information friction, so interest rates are fairly low over a large range of investment sizes. This leads the two firms to make quite different investments, equating the relatively constant expected marginal cost with their quite expected marginal products. When the enforcement cost is high (panel b), the interest rate schedule become more sensitive to the effects of the information friction, so interest rates are higher over a larger range of investments. This leads the two firms to make investments that are closer together; the marginal cost of investing becomes prohibitive for firm 2 more quickly than before.

These two examples, of course, don't fully capture the extent of the misallocation that occurs in the model. In general, firms that have high expected productivities are too small relative to a perfect-information

Figure 6: Illustration of misallocation between firms with the same c



benchmark, while those that have low expected productivities are too large. Figure 7 illustrates this point by plotting the equilibrium capital allocations for all firms in the economy about whom lenders receive the mean signal ($\eta = 0$, i.e., $c = b$), relative to the allocations these firms would receive in a perfect-information economy. The red line plots these values for an economy with strong contract enforcement ($\phi < 1$). Firms with very low b are larger than they would be in a perfect information setting, while firms with higher b are smaller. As b grows, firms become smaller relative to their perfect-information sizes – the information friction affects high- b firms more. The blue line plots the same values for an economy with weak contract enforcement ($\phi = 1$), and we see that the pattern becomes more pronounced. Low- b firms are larger relative to the perfect-information world, while high- b firms are smaller.

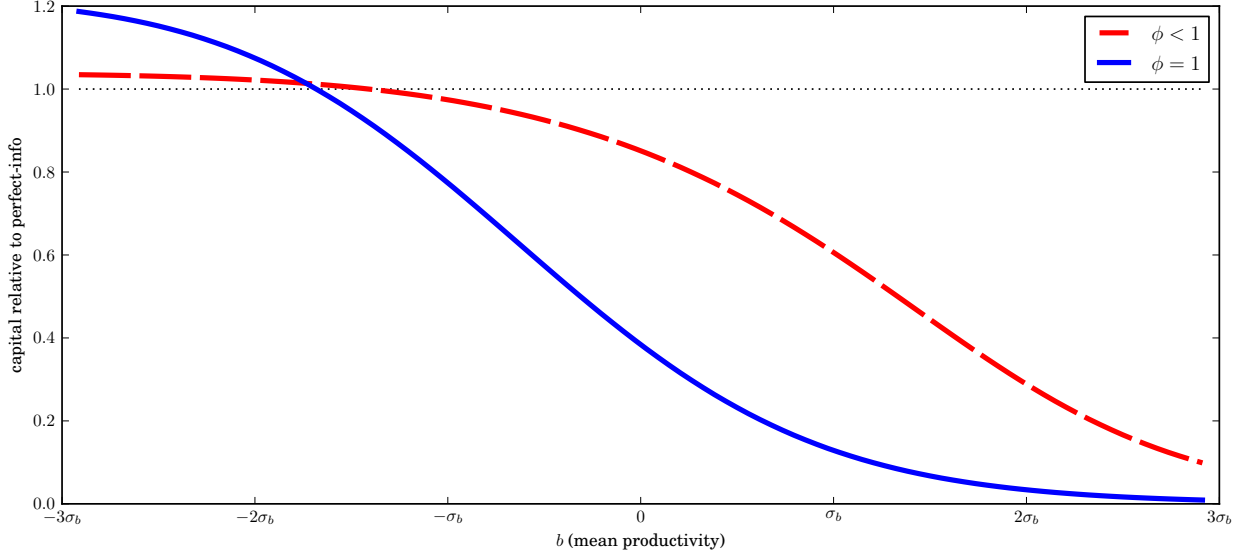
As mentioned above, in frictionless benchmark models all firms typically have the same marginal products; in the absence of any distortions all firms choose factor inputs to set marginal products equal to factor rental rates. In the perfect-information version of my model, this is not quite true due to the risk in firms' production technologies implied by the model's timing. However, a very similar result holds:

Proposition 8.

- (i) If $\sigma_\eta = 0$ then all firms have the same expected marginal product of capital.
- (ii) If $\sigma_\eta > 0$ there is dispersion in the expected marginal product of capital.

The intuition for this result is that the interest rate schedules $r(k, c)$ are functions only of $(1 - \alpha) \log(k) - \mu_{a|c}$

Figure 7: Illustration of misallocation among firms with mean lender signals ($\eta = 0$)



and when $\sigma_\eta = 0$ we have $\mu_{a|c} = \mu_{a|b}$. In this case we can write the firm's first-order condition (13) as

$$\alpha \exp \left[- \left((1 - \alpha) \log(k(b)) - \mu_{a|b} \right) + \frac{\sigma_{a|b}^2}{2} \right] Q((1 - \alpha) \log(k(b)) - \mu_{a|b}) = R((1 - \alpha) \log(k(b)) - \mu_{a|b}) \quad (25)$$

where Q and R are functions of $(1 - \alpha) \log(k(b)) - \mu_{a|b}$. Hence all firms choose the same value of $(1 - \alpha) \log(k(b)) - \mu_{a|b}$. Since c is redundant in this case, I've used $k(b)$ to denote equilibrium capital allocations to economize on notation. The left-hand side of the equation above is the expected marginal product, so that is equalized across firms as well. Appendix A provides a formal proof. This proposition is the model's version of the first welfare theorem. In the appendix I show that a social planner who shares firms' uncertainty about their productivities a will also choose an allocation such that all firms have the same expected marginal products. When $\sigma_\eta > 0$, $\mu_{a|c}$ will generally be different from $\mu_{a|b}$ so the above result no longer holds, which means that there will be some dispersion of expected marginal products across firms. As one might expect, in numerical experiments the amount of dispersion increases with σ_η , and increases more for higher values of ϕ .

4.5 Numerical exercise

In this section I parameterize the two-period model described above to show directly that the misallocation the imperfect information friction generates reduces TFP, and that this effect is larger when contracts are more costly to enforce. I first pick parameters so that the perfect-information ($\sigma_\eta = 0$) version of the model

matches several U.S. data moments, then vary the amount of noise in the model while holding all other parameters fixed. We see that aggregate TFP falls as the amount of noise increases, falling faster when contracts are harder to enforce. I also show that dispersion in the marginal product of capital is increasing in the amount of noise, and that that effect is once again more pronounced when contracts are harder to enforce. Finally, I show that noisier signals reduce the correlation between capital and productivity and increase the correlation between marginal product and productivity at the firm level, both of which indicate that more productive firms are more constrained.

These results serve to complement the quantitative results in the next section, which focus on a binary comparison between perfect information and no-information economies to assess the maximum impact of the model's information friction. The quantitative model is computationally intractable at intermediate levels of the information friction, but the stylized model of this section is not. This numerical exercise allows us to examine what happens as we begin to move away from the perfect-information setting, gaining some insight into how quickly the effects of the information friction begin to take hold.

Table 2 below lists the parameter values I use in this exercise. The returns to scale and recovery rate take similar values as they do in the calibration of the quantitative model in the next section. I choose the other parameters σ_a and σ_ε so that the firm size distribution and exit rate match the quantitative model's moments. Given the stylized nature of this version of the model, I don't interpret this as a true calibration. Nevertheless, the upper bound on the TFP losses in this numerical exercise are very close to what I find in the quantitative model.

Table 2: Parameterization

Parameter	Value
α	0.65
ϕ	0.75
σ_a	0.97
σ_ε	0.20

4.5.1 Results

Holding fixed the parameters calibrated above I vary σ_η , the amount of noise in the lenders' signals, and report the quantitative magnitudes of the changes in the variables in question. To study the extent to which limited enforcement compounds the effects of the information friction, I repeat the exercise for $\phi \in$

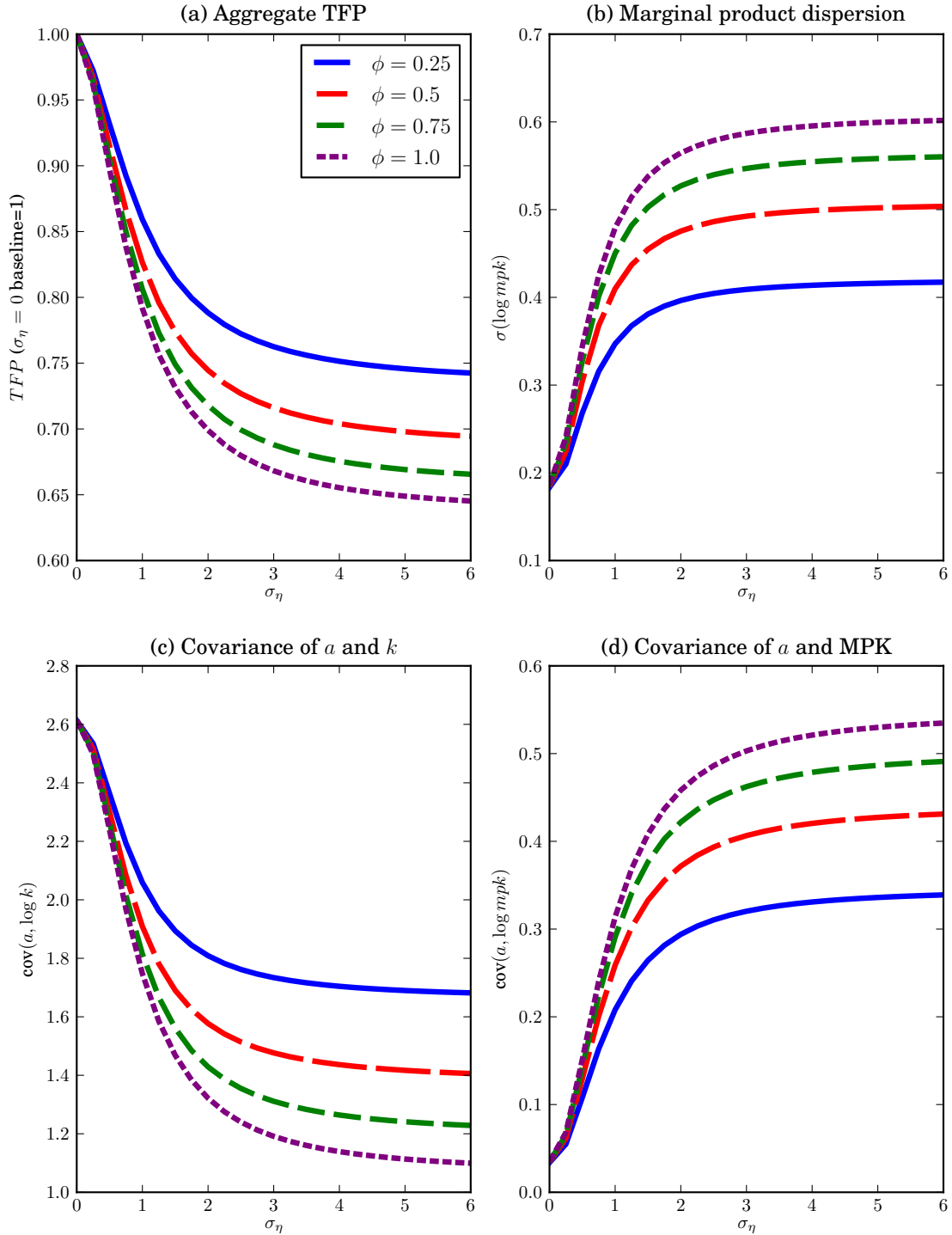
$\{0, 0.25, 0.5\}$. Note that there is a finite limit to effects of increasing the noise; lenders offer contracts even when $\sigma_\eta = \infty$. In this extreme case, lenders view all firms identically so all firms face the same interest rate schedules.

Figure 8 reports the results of the exercise. In panel (a) I plot total factor productivity (normalized to one in the $\sigma_\eta = 0$ benchmark) against the noise parameter σ_η . In the baseline model with $\phi = 0.75$ (the dark blue line), as σ_η becomes large TFP falls by approximately 25% relative to the no-noise benchmark. This number is comparable to results of similar exercises in quantitative studies on finance and misallocation that focus on limited enforcement frictions. On the high end, Amaral and Quintin (2010) report that limited enforcement frictions can cause TFP to fall by as much as 60%. On the low end, Midrigan and Xu (2010) find only 5-7%. Other studies report numbers between these two extremes; Buera, Kaboski, and Shin (2011) for example find that limited enforcement can cause TFP to fall by as much as 35%. In other words, the quantitative effects of the information friction I study is indeed economically significant when compared to the reported effects of limited enforcement frictions. The other lines in the graph plot results for lower values of the recovery rate ϕ . TFP falls by up to 30% when $\phi = 0.5$, by up to 32.5% when $\phi = 0.25$, and up to 35% when $\phi = 0$. Hence limited enforcement exacerbates the effects of the noise in my model. Put another way, limited enforcement and noisy financial information are complements in terms of their effects on aggregate TFP. In panels (b) - (d) I plot three distributional measures of misallocation against σ_η . Panel (b) shows the effects of increasing noise on the covariance of productivity and capital. When capital is allocated efficiently, the most productive firms have the most capital so this covariance should be high. A drop in this covariance indicates a decrease in the efficiency of the capital allocation. The figure indicates that varying σ_η causes this covariance to fall by 30% in the baseline case, by 46% when $\phi = 0.5$, by 55% when $\phi = 0.25$, and by 60% when $\phi = 0$.

Panel (c) shows the effects of the noisy information on the dispersion of the marginal product of capital, measured here as the standard deviation of the log of the marginal product, the same misallocation measure employed by Hsieh and Klenow (2009). While this model exhibits equality of expected rather than realized marginal products in the noiseless benchmark case, I choose to use the realized expected marginal product to facilitate comparison with the data. However, the quantitative effects on expected marginal product dispersion are similar. This measure increases by up to 0.2 in the benchmark parameterization, by 0.3 when $\phi = 0.5$, by 0.35 when $\phi = 0.25$, and 0.4 when $\phi = 0$. In comparison, Hsieh and Klenow (2009) report that, compared to the U.S., this measure is 0.29 and 0.14 higher in China and India respectively.⁷

⁷Note however that Hsieh and Klenow (2009) report that the standard deviation of log TFPR is 0.45 in the U.S. as compared to 0.2 in my noiseless benchmark. This difference is not surprising given that I choose not to target the marginal product distribution in my calibration. When I target this figure of 0.45 directly instead of the average default rate, the quantitative effects of increasing σ_η on aggregate TFP increase substantially.

Figure 8: Numerical results: TFP and misallocation in a two-period model



Panel (d) shows the effects of σ_η on the covariance between productivity and marginal product. This misallocation measure tells us how much more constrained (in terms of how much capital they are able to

employ) productive firms are than unproductive ones. This measure increases by up to 0.25 in the benchmark parameterization, by 0.35 when $\phi = 0.5$, by 0.4 when $\phi = 0.25$, and 0.45 when $\phi = 0$. Put together, the results in panels (b) - (d) indicate that the information friction I study in this paper has large quantitative impact on distributional measures of misallocation commonly employed in the literature. Moreover, just as with aggregate TFP, limited enforcement magnifies the information friction's effects.

The results described above imply that noisy financial information has quantitatively and economically significant effects on misallocation and aggregate productivity. We've also seen that limited enforcement exacerbates the effects of noisy information, which means that it's important to take seriously the notion that financial development has multiple, interrelated aspects. However, it's important to point out that none of the results listed thus far are unique implications of noisy information; all of the changes in distributional measures of misallocation reported above can be rationalized by limited enforcement. A key feature of all limited enforcement models is that productive firms are more constrained than unproductive ones relative to a frictionless benchmark. This typically implies a drop in the covariance of productivity and factor allocations, an increase in the dispersion of marginal products and an increase in the productivity–marginal product covariance.

5 Dynamic model

The two-period model studied in section 4 above illustrates why noisy information causes misallocation of resources across firms that reduces aggregate TFP. The basic idea is straightforward: when borrowing constraints and interest rates that lenders offer to firms are inconsistent with firm's own signals about their productivities, many firms that are likely to be productive face tight borrowing constraints and high interest rates, while other firms that are likely to be unproductive face loose borrowing constraints and low interest rates. The numerical exercise conducted using this model indicates that this information friction is quantitatively important for aggregate TFP. The static nature of this model, however, prevents firms from accumulating assets over time. Other papers in the literature on finance and misallocation like Moll (2012) argue that asset accumulation financed in part by internal funds ("self-financing") can undo the misallocation caused by financial frictions. In this section I develop a dynamic extension of my model that allows firms to build up capital gradually over time. Building on earlier models of firm dynamics like Hopenhayn (1992) and Cooley and Quadrini (2001), I incorporate default following more recent work by Arellano, Bai, and Zhang (2011) and D'Erasmus and Boedo (2012).

I use a calibrated version of the model to conduct a similar quantitative exercise to that performed above, comparing the perfect-information model with the a version in which lenders have no firm-specific

information (i.e. signals are infinitely noisy) to assess the maximum impact of the information friction on TFP, GDP per capita, and dispersion in the marginal product of capital. I do this exercise for the baseline calibration in which the contract enforcement parameters are for the U.S. and an alternative scenario in which they are low-income country averages to analyze the interaction between contract enforcement and imperfect information.

I begin this section by describing the perfect-information version of the model and my calibration of this model to U.S. data. Next I discuss my approach for incorporating a similar information friction to the one I used in the two-period model. I then assess the impact of this friction on aggregate TFP and distributional measures of misallocation as I did with the two-period model.

5.1 Perfect information

Time is discrete and there is no aggregate uncertainty. I restrict my attention to stationary equilibria in which all aggregate variables are constant. There is a large number of identical households with discount factor β and a unit measure of firms of changing composition: each period some firms exit and other new firms enter. As in the two-period model there is a large number of lenders. Each period in the model is split into two sub-periods, the production stage and the investment stage. Production takes place at the beginning of the period and investment takes place at the end.

Households own all firms and lenders in the economy. Households do not own capital directly, but they deposit savings with lenders that is used to finance loans to firms. Given current deposits B , the representative household's problem each period is to choose new deposits B' to maximize its utility subject to its budget constraint:

$$W(D) = \max_{D'} \{u(C) + \beta W(D')\} \quad (26)$$

subject to

$$C + D' = \Pi + (1 + r^*)D \quad (27)$$

where Π represents the aggregate sum of dividends issued by firms. As in the two-period model, lenders earn zero profits in expectation on all loans there are zero profits earned in the aggregate by all of the economy's lenders put together. Because there is no aggregate uncertainty, in a stationary equilibrium the gross risk-free interest rate must be equal to the inverse of the households' discount factor: $\beta = 1/(1 + r^*)$. As a consequence, any level of deposits can be sustained in equilibrium. I therefore assume that deposits simply equal the total amount loaned to firms. The household's problem plays no role in what follows, but I include it for completeness to flesh out the explicit general equilibrium of the model.

Firms produce output using capital and labor in a decreasing-returns technology:

$$y = e^{b+\varepsilon} k^\alpha n^\theta, \quad 0 < \alpha + \theta < 1 \quad (28)$$

Here, b is a firm's mean productivity (which may change throughout its life), ε is an i.i.d. shock drawn from a distribution $F(\varepsilon)$. In addition to the productivity shock ε , I assume that firms must also pay a i.i.d. fixed cost f drawn from a distribution $H(f)$. The fixed cost is paid in units of output. Firms enter the production stage with mean productivities b , productivity shocks ε , capital k and repayment due to lenders ℓ_R . I assume that firms have limited liability and cannot issue new equity, which means that, just like in the static model, firms that cannot generate sufficient revenues to repay lenders must default. Firms that do not default hire workers n at wage w , rebate profits $y - wn - f - \ell_R$ to their owners, and continue on to the investment stage. Surviving firms retain their mean productivities b with probability π and draw new mean productivities b' from a distribution $G(b')$ with probability $1 - \pi$.

In the investment stage, incumbent firms (those that survived production stage) may choose to invest to augment their existing capital stocks. I assume that this adjustment is costly: incumbent firms must pay a per-unit cost $\psi(k, k')$ to adjust their capital from k to k' . The assumption that firms pay out all revenues net of loan repayment as dividends implies that firms have no cash on hand at the investment stage, so they must take out a loan of $\ell = k' - (1 - \delta)k + \psi(k, k')$ to finance their investment and adjustment costs. Each loan ℓ implies a repayment amount ℓ_R , described below, which the firm must pay in next period's production stage if it can generate sufficient revenues. In this stage a measure of new firms enters and also makes investment decisions. For simplicity, I assume that entry is costless and that the measure of new entrants is equal to the measure of firms that exited in the production stage (through either default or exogenous death). New entrants draw initial mean productivities b from the same distribution $G(b)$ introduced above. New entrants have no initial capital or other assets and must also borrow from lenders to finance their initial investment. I assume that new entrants do not pay adjustment costs. They need only borrow to finance their initial choices of capital k' , hence for new entrants the loan size is simply $\ell = k'$. Once investment decisions are made, the period ends and the next period's production stage follows.

As in the static model, the fact that some firms default in equilibrium implies that the repayment amounts ℓ_R must compensate lenders for default risk. I assume that firms can use their capital stocks as collateral. If a firm which has taken out a loan of size ℓ to attain a capital stock of k' defaults, one of two things may occur. If the firm's capital stock k' is larger than or equal to its loan ℓ , the firm liquidates its capital and the lender receives $\lambda\ell$, where λ is recovery cost. If $k' < \ell$, the lender takes possession of the capital stock but must pay a per-unit cost of ϕ to do so. Hence the lender receives a payout of $\min\{\lambda\ell, (1 - \phi)k'\}$

if the firm defaults. This default contingency structure is almost identical to that employed in D'Erasmus and Boedo (2012). The repayment amount ℓ_R associated with a loan of size ℓ to a firm that attains capital k compensates for the risk the lender faces. The contracting objects that lenders offer are loan sets $L(b, k')$ and interest rate schedules $\ell_R(b, k', \cdot) : L(b, k') \rightarrow \mathbb{R}_+$. These objects now depend on the firm's next-period capital stock in addition to the loan size.

5.1.1 Firm's problem

Incumbent firms enter the first sub-period with mean productivity b , capital k , productivity shock ε , fixed cost shock f , and repayment amount ℓ_R . At this stage, if the firm can generate sufficient profits to repay ℓ_R , it hires labor n , produces output $y = e^{b+\varepsilon}k^\alpha n^\theta$, rebates profits $y - wn - f - \ell_R$, and continues on to the second sub-period. As mentioned above, firms cannot issue new equity, so if the firm's productivity shock ε is too low to satisfy this constraint, the firm must default and shut down. With probability $1 - \pi$, surviving firms draw a new mean productivity b' from the distribution $G(b)$. Let $\tilde{V}(b, k, \ell_R, \varepsilon, f)$ denote the value of the firm's problem at this stage, given by

$$\tilde{V}(b, k, \ell_R, \varepsilon, f) = \begin{cases} \pi(b, k, \varepsilon) - f - \ell_R + \pi V(b, k) + (1 - \pi) \int_{\mathbb{B}} V(b', k) dG(b') & \text{if } \pi(b, k, \varepsilon) - f \geq \ell_R \\ 0 & \text{otherwise} \end{cases} \quad (29)$$

where $\pi(b, k, \varepsilon)$, the firm's variable profit, is given by

$$\pi(b, k, \varepsilon) = \max_n \left\{ e^{b+\varepsilon}k^\alpha n^\theta - wn \right\} \quad (30)$$

and $V(b, k)$ denotes the value of the firm's problem at the investment stage. Let $g_d(b, k, \ell_R, \varepsilon)$ denote the default policy, given by

$$g_d(b, k, \ell_R, \varepsilon) = \begin{cases} 0 & \text{if } \pi(b, k, \varepsilon) \geq \ell_R + f \\ 1 & \text{otherwise} \end{cases} \quad (31)$$

This policy is characterized by a similar cutoff productivity shock $\underline{\varepsilon}(b, k, \ell_R, f)$ to the one in the static model, which solves

$$\pi(b, k, \underline{\varepsilon}(b, k, \ell_R, f)) = \ell_R + f \quad (32)$$

Note that this cutoff depends on the fixed cost shock – when the firm must pay a positive fixed cost it requires a higher productivity to stay in business.

Incumbent firms enter the investment stage with mean productivity b and initial capital stock k . At this

stage, an incumbent's problem is to choose investment x to maximize its discounted value next period. The firm must borrow from lenders to finance this investment. In particular, the firm takes out a loan of size $\ell = x + \psi(k, k')$, where $\psi(k, k')$ is the cost of adjusting the firm's capital stock from the current value k to next-period's value k' . This loan implies a repayment amount $\ell_R(b, k', \ell)$. The problem of an incumbent firm with mean productivity b , lender signal c and capital stock k in the first sub-period is to choose investment x to maximize

$$V(b, k) = \max_x \left\{ \mathbb{1}_{\{\ell < 0\}} \ell + \beta \int \tilde{V}(b, k', \ell_R(b, k', \ell), \varepsilon', f) dF(\varepsilon') dH(f) \right\} \quad (33)$$

subject to

$$k' = (1 - \delta)k + x \quad (34)$$

$$\ell = x + \psi(k', k) \quad (35)$$

$$\ell \in L(b, k') \quad (36)$$

The presence of the discount factor β represents the fact that the investment decision occurs at the end of the period, while revenues generated by the additional capital occur in the next period (if the firm does not default). I allow firms to undertake negative investment at this stage, although they still incur adjustment costs if they do so. Any positive cash flow from sales of capital (i.e. a negative loan) is distributed as dividends. Let $g_k(b, k)$ denote the optimal policy for capital accumulation. Note that this is sufficient to determine the firm's labor demand (conditional on producing) $g_n(b, k, \varepsilon)$, its repayment amount $g_{\ell_R}(b, k)$. It is also useful to define the firm's default cutoff policy $g_{\underline{\varepsilon}}(b, k, f)$:

$$g_{\underline{\varepsilon}}(b, k, f) = \underline{\varepsilon}(b, g_k(b, k), g_{\ell_R}(b, k), f) \quad (37)$$

This policy also depends on the fixed cost shock.

New entrants are born in the investment stage, after default decisions take place but before continuing incumbents make investment decisions. This timing ensures that all firms' investment decisions occur simultaneously. Let μ denote the mass of new entrants. New entrants draw initial mean productivities b from the distribution $G(b)$. New entrants are born with no capital and no debt. I assume that new entrants pay no adjustment costs on their initial choice of capital, which implies that a new entrant's loan is simply equal to its choice of initial capital. As we will see when I extend the model to an environment with noisy information, this assumption minimizes the effect of the noisy information problem on new entrants. This

choice biases my quantitative results about the impact of the noisy information friction downward. The problem of a new entrant is to choose capital stock k' to maximize its discounted value next period. This problem takes the same form as the investment problem of an incumbent firm:

$$V_e(b) = \max_{k'} \left\{ \beta \int \tilde{V}(b, k', \ell_R(b, k', \ell), \varepsilon', f) dF(\varepsilon') dH(f) \right\} \quad (38)$$

subject to

$$\ell = k' \quad (39)$$

$$\ell \in L(b, k') \quad (40)$$

Let $g_{e,k}(b)$, $g_{e,n}(b, \varepsilon)$, $g_{e,\ell_R}(b)$, and $g_{e,\varepsilon}(b, f)$ denote the optimal entrant policies for capital accumulation, labor demand, repayment amount, and cutoff respectively.

5.1.2 Lender's problem

As in the static model presented in section 4, lenders are perfectly competitive so they must break even in expectation on every possible loan they offer to each firm. In the perfect-information environment, lenders know all firms' mean productivities b with certainty. The break-even condition that pins down repayment amounts is

$$(1 + r^*)\ell = (1 - p_b(k', \ell_R(b, k', \ell))) \ell_R(b, k', \ell) + p_b(k', \ell_R(b, k', \ell)) \min\{\lambda\ell, (1 - \phi)k'\} \quad (41)$$

This equation is very similar to the one in the static model. The only difference is that firm's capital stocks will in general be different from the amount they borrow from lenders, which changes lenders' payoffs when firms default. As a consequence, repayment amounts depend on installed capital in addition to the amount loaned. Since lenders know firm's mean productivities with certainty in the perfect-information version of the model, the default probability is simply

$$p_b(k', \ell_R) = \int F(\varepsilon(b, k', \ell_R, f)) dH(f) \quad (42)$$

5.1.3 Aggregation and equilibrium

The structure and timing of the model imply that the distribution of incumbent firms over mean productivities and capital at the beginning of the investment stage and the measure of new entrants are sufficient

statistics for all aggregate variables, including next period's distribution and entry rate. Let B and K denote the set of possible mean productivities and capital stocks that firms can have, and let \mathcal{B} and \mathcal{K} denote typical subsets of the sets. Let Λ denote the distribution over $B \times K$ and recall the μ is the measure of new entrants. The law of motion for this distribution is a map T , given by

$$\begin{aligned} T_\Lambda(\Lambda, \mu)(\mathcal{B} \times \mathcal{K}) &= \pi \int_B \int_K \int_F \int_{g_\varepsilon(b,k,f)}^\infty \mathbb{1}_{\{g_k(b,k) \in \mathcal{K}\}} dF(\varepsilon) dH(f) d\Lambda(b,k) \\ &\quad + (1 - \pi) \int_B \int_{B \times K} \int_F \int_{g_\varepsilon(b,k,f)}^\infty \mathbb{1}_{\{g_k(b,k) \in \mathcal{K}\}} dF(\varepsilon) dH(f) d\Lambda(b,k) dG(b') \\ &\quad + \mu \int_B \int_F \int_{g_{e,\varepsilon}(b,f)}^\infty \mathbb{1}_{\{g_{e,k}(b) \in \mathcal{K}\}} dF(\varepsilon) dH(f) dG(b) \end{aligned} \quad (43)$$

The first line represents incumbent firms who ended last period with mean productivities in \mathcal{B} who chose new capital stocks in \mathcal{K} , did not exit, and retained their old mean productivities. The second line represents surviving incumbent firms that chose new capital stocks in \mathcal{K} that drew new mean productivities in \mathcal{B} before reaching the investment stage. The last and final line represents new entrants that drew mean productivities in \mathcal{B} and chose initial capital stocks in \mathcal{K} .

The measure of firms that exit is given by a map T_μ :

$$T_\mu(\Lambda, \mu) = \int_{B \times K} \int_F \int_{-\infty}^{g_\varepsilon(b,k,f)} dF(\varepsilon) dH(f) d\Lambda(b,k) + \mu \int_B \int_F \int_{-\infty}^{g_{e,\varepsilon}(b,f)} dF(\varepsilon) dH(f) dG(b) \quad (44)$$

The first component represents incumbent firms that default, the second represents new entrants that default, and the third and fourth represent incumbents and new entrants that do not exit. In any stationary equilibrium the distribution Λ and the measure of entrants μ must be constant, i.e., $\Lambda = T_\Lambda(\Lambda, \mu)$ and $\mu = T_\mu(\Lambda, \mu)$.

The key aggregate variables are capital, labor, output, and the measure of firms that produce. The aggregate capital stock is⁸

$$K = \int_{B \times K} \int_F \int_{g_\varepsilon(b,k,f)}^\infty g_k(b,k) dF(\varepsilon) dH(f) d\Lambda(b,k) + \mu \int_B \int_F \int_{g_{e,\varepsilon}(b,f)}^\infty g_{e,k}(b) dF(\varepsilon) dH(f) dG(b) \quad (45)$$

The first line represents capital of non-defaulting incumbent firms while the second line represents capital of non-defaulting new entrants. Similarly, aggregate labor demand is

$$N = \int_{B \times K} \int_F \int_{g_\varepsilon(b,k,f)}^\infty g_n(b,k,\varepsilon) dF(\varepsilon) dH(f) d\Lambda(b,k) + \mu \int_B \int_F \int_{g_{e,\varepsilon}(b,f)}^\infty g_{e,n}(b,\varepsilon) dF(\varepsilon) dH(f) dG(b) \quad (46)$$

⁸To be precise, this is the aggregate stock of capital used in production. defaulting firms do not produce and give up their capital to lenders or the firms' owners, depending upon whether their capital exceeds their debts.

I normalize the aggregate supply of labor to one. Aggregate GDP is

$$\begin{aligned}
Y = & \int_{B \times K} \int_F \int_{g_{\underline{\varepsilon}}(b,k,f)}^{\infty} \left(e^{b+\varepsilon} g_k(b,k)^\alpha g_n(b,k,\varepsilon)^\theta - f \right) dF(\varepsilon) dH(f) d\Lambda(b,k) \\
& + \mu \int_B \int_F \int_{g_{e,\underline{\varepsilon}}(b,f)}^{\infty} \left(e^{b+\varepsilon} g_{e,k}(b,k)^\alpha g_{e,n}(b,\varepsilon)^\theta - f \right) dF(\varepsilon) dH(f) dG(b)
\end{aligned} \tag{47}$$

The measure of firms that produce is

$$M = \int_{B \times K} \int_F \int_{g_{\underline{\varepsilon}}(b,k,f)}^{\infty} dF(\varepsilon) dH(f) d\Lambda(b,k) + \mu \int_B \int_F \int_{g_{e,\underline{\varepsilon}}(b,f)}^{\infty} dF(\varepsilon) dH(f) dG(b) \tag{48}$$

Since the aggregate capital stock is constant in a stationary equilibrium, aggregate investment is simply equal to the depreciation of that stock. New entrants and incumbents that undertake investment above and beyond depreciation on their own capital, but this is offset by liquidation of exiting firms and negative investment undertaken by large firms that draw low mean productivities. Aggregate consumption is equal to aggregate output less investment and adjustment costs:

$$C = Y - \delta K - \int_{B \times K} \psi(k, g_k(b,k)) d\Lambda(b,k) \tag{49}$$

With all aggregate variables defined, I proceed to formally define a stationary equilibrium:

Definition 2. *A stationary equilibrium is*

- Real interest rate r^* and wage w
- Distribution Λ and entry rate μ
- Value functions $V(b,k)$ and $\tilde{V}(b,k, \ell_R, \varepsilon)$
- Policy functions $g_k(b,k)$, $g_n(b,k)$, $g_{\ell_R}(b,k)$ and $g_{\underline{\varepsilon}}(b,k)$ for incumbent firms
- Policy functions $g_{e,k}(b)$, $g_{e,n}(B)$, $g_{e,\ell_R}(b)$ and $g_{e,\underline{\varepsilon}}(b)$ for new entrants
- Loan sets $L(b,k')$ and repayment amounts $\ell_R(b, \cdot, k') : L(b,k') \rightarrow \mathbb{R}_+$

such that (i) incumbent firms, new entrants and lenders solve their respective problems, (ii) the market for labor clears, and (iii) the distribution and default rate are stationary: $\Lambda = T\Lambda$ and $\mu = T\mu$.

5.2 Calibration

I calibrate the perfect-information version of the dynamic model to U.S. data. This is a common approach in the literature on financial frictions and misallocation (see, for example, Amaral and Quintin (2010); Moll

(2012); Buera, Kaboski, and Shin (2011)). The implicit assumption in this approach is that lenders have perfect information about borrowers' default risk in the United States. While U.S. financial markets are certainly not perfect (especially in light of the recent financial crisis), the U.S. gets the maximum possible rating in the *Doing Business* depth of credit information ranking I used in the empirical section above.

I set the length of a period in the model to one year. I take the representative household's discount factor and the factor share and contract enforcement parameters from the literature. I choose the discount factor β so that the equilibrium risk-free rate is 4% per year. I take the capital and labor shares in production from Restuccia and Rogerson (2008), setting α to 0.283 and θ to 0.567. This implies a return to scale of 0.85. I choose a standard quadratic form for the adjustment cost: $\psi(k, k') = \psi(x/k)^2 k/2$. I set ψ to 0.01, an intermediate value in the literature, which reports calibrated or estimated values from 0.043 (Cooper and Haltiwanger, 2006) to 0.006 (Arellano, Bai, and Zhang, 2011). In addition, I assume that firms cannot sell capital at full price; negative investment incurs a 10% discount, which is also in line with the literature. I denote the resale price of capital by p_s . I choose the enforcement parameters, ϕ and λ , from D'Erasmus and Boedo (2012), who in turn take them from *Doing Business*. ϕ , the cost associated with bankruptcy proceedings, is 0.015. λ , the recovery rate, is 0.77.

I set the remaining parameters to target data moments. In general, these parameters affect the target moments in a nonlinear, non-independent manner. For the sake of clarity, however, I list each parameter's intended target (despite the fact that other parameters affect it as well) below. I set the depreciation rate δ so that depreciation to GDP matches the 11% figure reported in the NIPA tables. I assume that the distribution $G(b)$ of firms' mean productivities is Pareto with tail parameter γ_b , which I choose to match the coefficient of variation of employment (which I take from the empirical CDF used by Restuccia and Rogerson (2008)). The calibrated value of γ_b is 2.25. In solving the model numerically I discretize $G(b)$ with 21 values. I choose the probability of switching mean productivities, π , and the variance of the i.i.d. shocks σ_ϵ^2 so that coefficient estimates from an AR(1) regression on firm-level TFP (conditional on survival) match those reported by Foster, Haltiwanger, and Syverson (2008). The calibrated value of π and σ_ϵ are 0.95 and 0.2, respectively. I assume that the i.i.d. fixed cost shock f can take three values: 0, $f > 0$, and ∞ (excuse the abuse of notation). The positive and infinite values occur with probabilities p_f and p_∞ respectively. I jointly set the size of the finite fixed cost f and the two fixed cost probabilities p_f and p_∞ so that the model matches the overall 10% exit rate, the 4% exit rate of firms in the top 5% of the employment distribution, and the 14% exit rate of firms in the bottom 50% of the employment distribution reported by Boedo and Mukoyama (2012). Table 3 below lists all of the model's parameters and target moments or sources in the literature.

Table 3: Calibrated parameters

Parameter	Value	Source/target statistic	Data	Model
<i>Chosen parameters</i>				
β	0.9615	Real interest rate	4%	4%
α	0.283	Restuccia and Rogerson (2008)	–	–
θ	0.567	Restuccia and Rogerson (2008)	–	–
ψ	0.01	Multiple studies	–	–
p_s	0.9	Cooper and Haltiwanger (2006)	–	–
ϕ	0.07	D’Erasmus and Boedo (2012)	–	–
λ	0.77	D’Erasmus and Boedo (2012)	–	–
<i>Calibrated parameters</i>				
δ	0.08	Depreciation/GDP	11%	11.4%
γ_b	2.5	Coeff. variation, emp.	5.95	5.88
π	0.95	TFP persistence	0.75	0.75
σ_ε	0.2	Variance of TFP residuals	0.3	0.27
f	0.007	Exit rate, all firms	10%	10.8%
p_f	0.12	Exit rate, bottom 50% emp.	14%	12.4%
p_∞	0.03	Exit rate, top 5% emp.	4%	5.0%

5.3 Adding information frictions

In this dynamic environment, incorporating a similar noisy information friction to the one in the two-period model is more challenging. Several questions arise: Can lenders learn about firms’ mean productivities over time? What happens when a firm draws a new mean productivity? As a first pass, we can sidestep these questions by studying the worst-case scenario when there is infinite variance in the noise in lenders’ signals so that there is no scope for learning and the values of lenders’ signals are meaningless. This exercise is in fact quite useful since it informs us about the maximum effect of the noisy information friction on TFP.

When lenders’ signals contain no information, the probability that an incumbent firm defaults from a lender’s perspective is now

$$\mathbb{E} [p_b(k', \ell_R) | k] = \int_B p_b(k', \ell_R) d\Lambda(b|k) \quad (50)$$

Lenders offer the same loan sets $L(k, k')$ and repayment function $\ell_R(\cdot, k, k') : L(k, k') \rightarrow \mathbb{R}_+$ to all incumbent firms. Notice that these objects no longer depend on the firm’s signal b . The definition of equilibrium must

be modified only slightly since the firm’s problem remains unchanged. I maintain the assumption that lenders cannot infer firms’ types from firms’ investment decisions. For new entrants, the probability of default from a lender’s perspective is similar:

$$\mathbb{E} [p_b(k', \ell_R)] = \int_B p_b(k', \ell_R) dG(b) \tag{51}$$

Lenders offer the same loan sets $L_e(k')$ and repayment function $\ell_{e,R}(\cdot, k') : L_e(k') \rightarrow \mathbb{R}_+$ to all new entrants.

Table 4 below compares key model variables from the infinite-noise version to the perfect-information benchmark. The second column lists the results of the comparison for the baseline calibration. The third column lists the results for what I call the “weak enforcement” scenario in which the parameters ϕ and λ are set to 0.15 and 0 respectively, the values used by D’Erasmus and Boedo (2012) which correspond to *Doing Business* average measures for low-income countries. We see that TFP falls by 23% in the baseline calibration and by 32% in the weak enforcement scenario. GDP per capita falls by 57% in the baseline calibration and by 67% in the weak enforcement scenario.

Table 4: Imperfect information model relative to perfect-information benchmark

Variable	Baseline enforcement	Weak enforcement
TFP	0.77	0.68
GDP per capita	0.43	0.33

Statistics for perfect-information benchmark normalized to one for each column separately.

The first takeaway from this exercise is that even when we allow for firms to accumulate capital over time and add other quantitative ingredients, the information friction in my model still has a quantitatively significant effect on TFP. The second takeaway is that, just as in the two-period model, weak contract enforcement exacerbates the effects of imperfect information about default risk, causing TFP to fall by an additional 10% and GDP per capita by an additional 9%. The third takeaway is that by allowing for capital accumulation, the two frictions in the model affect GDP per capita through their impact on the aggregate capital stock in addition to TFP. As a consequence, GDP per capita falls by significantly more than TFP.

One would expect several of the model’s parameters to play a key role in driving these results. I focus my sensitivity analysis in three parameters that affect the ability of firms to build up capital over time. The probability of switching mean productivities, π , governs the probability that a highly productive firm will retain its mean productivity long enough to build up its capital stock to the level that would prevail in the absence of information frictions. The adjustment cost parameter, γ , governs the speed at which such

a firm can build up its capital stock. Finally, the probability of an infinite fixed cost, p_e , governs that rate at which even the most productive firms exit the market. Table 5 below lists the results of this sensitivity analysis. We see that the key results are in fact robust to changes in these parameters: the magnitude of the drops in TFP and GDP per capita that the information friction causes in both enforcement scenarios change little when we vary these three parameters. Most importantly, the quantitative impact of the interaction between the information and enforcement frictions remains roughly the same: under all three alternative parameterizations, the information friction causes TFP and GDP per capita to fall by approximately 10% more in the weak enforcement scenario.

Table 5: Imperfect information model relative to perfect-information benchmark under alternative parameterizations

Variable	Baseline enforcement	Weak enforcement
<i>(a) $\pi = 0.75$</i>		
TFP	0.82	0.74
GDP per capita	0.54	0.4
<i>(b) $\gamma = 0.05$</i>		
TFP	0.76	0.66
GDP per capita	0.42	0.31
<i>(c) $p_e = 0.06$</i>		
TFP	0.76	0.69
GDP per capita	0.41	0.32

Statistics for perfect-information benchmark normalized to one for each column separately.

6 Conclusion

This paper studies the combined effects of imperfect information, in the form of uncertainty about borrowers' default risk, and limited contract enforcement on misallocation of resources and TFP. As motivation, I use cross-country data to show that depth of credit information is associated with increased TFP and that this effect is larger in countries with weaker contract enforcement. I then build a model of a financial mar-

ket in which borrowers are heterogeneous in their default risk, but lenders only have imperfect information (noisy signals) about each individual borrower and must pay a cost to recover from borrowers who default. Using their knowledge about the distribution of default probabilities, lenders use what information they have to estimate each borrower's default probability and use these estimates to set interest rates on loans. This simple model illustrates how the cost of contract enforcement makes the effect of the imperfect information friction larger. Each of these frictions has one direct effect. Limited enforcement reduces lenders' payoffs from defaulting borrowers, while the imperfect information friction in the model makes lenders' estimates of borrowers' default risk less accurate. Lenders must increasingly rely on increased interest rates to compensate for default risk when they get low payoffs when default occurs, so imperfect information affects interest rates more when the enforcement cost is high.

After using this simple model to illustrate the intuition behind the interaction between these two frictions, I embed the model in a production economy with heterogeneous firms that must borrow to finance investment to study how this intuition translates into misallocation. First, I use a stylized version of the model to prove several analytical results about the effects of imperfect information and limited enforcement on the allocation of capital to highlight the mechanism. I show that imperfect information increases the variation in capital allocations among firms of the same type and increases variation in the marginal product of capital, and that these effects are larger when enforcing contracts is more costly. I then extend the model to a quantitative, dynamic setting in which firms can accumulate capital over time, potentially mitigating the effects of these two frictions. I calibrate the perfect-information version of this model to U.S. data and use it to assess the quantitative impact on misallocation and TFP from introducing imperfect information. In the baseline parameterization for enforcement costs, I find that TFP falls by up to 24% relative to the perfect-information benchmark, while GDP per capita falls by up to 57%. When I use values associated with low-income countries for the enforcement cost parameters, I find that TFP and GDP per capita fall by up to 34% and 66%, respectively.

These results indicate that imperfect information about borrowers is more harmful for economic development and aggregate productivity in economies with poor contract enforcement. This implies that policies that reform financial markets along these dimensions are substitutes, in that reform along one dimension will have a larger effect on economic development than a subsequent reform along the other (Asturias, Hur, Kehoe, and Ruhl, 2012). The strength of contract enforcement is often seen as a deep-rooted institutional quality. Djankov, McLiesh, and Schleifer (2007) find that the origin of a country's legal system is an important determinant of its level of contract enforcement – contracts are typically easier to enforce in common-law countries than French civil law countries. As a consequence, policies that improve the credit information environment, like implementation of public credit bureaus, may act as substitutes for

improvements in contract enforcement and at the same time may be easier to implement.

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A Proofs

Lemma 1. *Let X and Y be two random variables such that $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$ and $Y|X = x \sim \mathcal{N}(mx, \sigma_{Y|X}^2)$. Then X and Y are jointly normal, i.e.,*

$$(X, Y) \sim \mathcal{N} \left(\begin{bmatrix} \mu_X \\ \mu_Y \end{bmatrix}, \begin{bmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{bmatrix} \right)$$

with $\mu_Y = m\mu_X$, $\sigma_Y^2 = \sigma_{Y|X}^2 + \sigma_X^2 m^2$ and $\rho = \frac{\sigma_X}{\sqrt{\sigma_{Y|X}^2 + \sigma_X^2 m^2}}$.

Proof of lemma 1. Let $f_X(x)$ and $f_Y(y|X = x)$ denote the unconditional density of X and the density of Y conditional on $X = x$ respectively. The joint distribution $f_{X,Y}(x, y)$ is equal to $f_X(x)f_Y(y|X = x)$. Using what we know about $f_X(x)$ and $f_Y(y|X = x)$, we can write this as

$$\begin{aligned} f_{X,Y}(x, y) &= \frac{1}{2\pi\sigma_X\sigma_{Y|X}} \exp \left\{ -\frac{(x - \mu_X)^2}{2\sigma_X^2} - \frac{(y - mx)^2}{2\sigma_{Y|X}^2} \right\} \\ &= \frac{1}{2\pi\sigma_X\sigma_{Y|X}} \exp \left\{ -\frac{(x - \mu_X)^2}{2\sigma_X^2} - \frac{y^2 - 2mxy + a^2x^2}{2\sigma_{Y|X}^2} \right\} \end{aligned}$$

If X and Y are jointly normal then

$$f_{X,Y}(x, y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp \left\{ -\frac{1}{2(1-\rho^2)} \left[\frac{(x - \mu_X)^2}{\sigma_X^2} + \frac{(y - \mu_Y)^2}{\sigma_Y^2} - \frac{2\rho(x - \mu_X)(y - \mu_Y)}{\sigma_X\sigma_Y} \right] \right\}$$

Moreover, if X and Y are jointly normal,

$$Y|X = x \sim \mathcal{N} \left(\mu_Y + \frac{\sigma_Y}{\sigma_X}\rho(x - \mu_X), (1 - \rho^2)\sigma_Y^2 \right)$$

Pick σ_Y, ρ and μ_Y so that $m = \frac{\sigma_Y}{\sigma_X} \rho, \sigma_{X|Y}^2 = (1 - \rho^2) \sigma_Y^2$ and $\mu_Y = \frac{\sigma_Y}{\sigma_X} \rho \mu_X$. Solving the first equation for ρ to get $\rho = \frac{\sigma_X}{\sigma_Y} m$ and substituting the result into the second yields $\sigma_Y^2 = \sigma_{Y|X}^2 + \sigma_X^2 m^2$. Then $\rho = \frac{\sigma_X}{\sqrt{\sigma_{Y|X}^2 + \sigma_X^2 m^2}}$ and $\mu_Y = m \mu_X$. Using these definitions in the formula for $f_{X,Y}(x, y)$ above and doing a bit of algebra we get

$$\begin{aligned} f_{X,Y}(x, y) &= \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp \left\{ -\frac{x^2 - 2\mu_X x + \mu_X^2}{2\sigma_X^2} - \frac{y^2 - 2\frac{\sigma_Y}{\sigma_X}\rho xy + \frac{\sigma_Y^2}{\sigma_X^2}\rho^2 x^2}{2(1-\rho^2)\sigma_Y^2} \right\} \\ &= \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp \left\{ -\frac{1}{2(1-\rho^2)} \left[\frac{(x - \mu_X)^2}{\sigma_X^2} + \frac{(y - \mu_Y)^2}{\sigma_Y^2} - \frac{2\rho(x - \mu_X)(y - \mu_Y)}{\sigma_X\sigma_Y} \right] \right\} \end{aligned}$$

Thus with σ_Y, ρ and μ_Y specified above X and Y are jointly normal. \square

Proof of proposition 1. I drop the i subscripts here for convenience. Using standard rules for Bayesian updating with normal distributions, conditional on c, b is normally distributed with mean $\mu_{b|c}$ and variance $\sigma_{b|c}^2$ given by

$$\mu_{b|c} = \left(\frac{\sigma_\eta^{-2}}{\sigma_b^{-2} + \sigma_\eta^{-2}} \right) c, \quad \sigma_{b|c}^2 = \frac{1}{\sigma_b^{-2} + \sigma_\eta^{-2}}$$

Recall that, conditional on b, a_i is normally distributed with mean $\mu_{a|b}$ and variance $\sigma_{a|b}^2$ given by

$$\mu_{a|b} = \left(\frac{\sigma_\varepsilon^{-2}}{\sigma_a^{-2} + \sigma_\varepsilon^{-2}} \right) b, \quad \sigma_{a|b}^2 = \frac{1}{\sigma_a^{-2} + \sigma_\varepsilon^{-2}}$$

Being a bit sloppy with the notation for random variables, we have $b|c \sim N(\mu_{b|c}, \sigma_{b|c}^2)$ and $a|b, c \sim N(mb, \sigma_{a|b}^2)$ where

$$m = \frac{\sigma_\varepsilon^{-2}}{\sigma_a^{-2} + \sigma_\varepsilon^{-2}}$$

By lemma 1 $a|c$ and $b|c$ are jointly normal with means $(\mu_{a|c}, \mu_{b|c})$ and covariance matrix diagonal $(\sigma_{b|c}^2, \sigma_{a|c}^2)$ where $\mu_{a|c} = m\mu_{b|c}$ and $\sigma_{a|c}^2 = \sigma_{a|b}^2 + m^2\sigma_{b|c}^2$. Using the rules of multivariate normal distributions, the marginal distribution of a conditional on c , i.e. the one with density $f(\cdot|c)$, is $a|c \sim N(\mu_{a|c}, \sigma_{a|c}^2)$. The desired result follows. \square

Proposition 9. Consider an economy with a fixed capital stock \bar{K} and a fixed mass M of firms indexed by i , each with productivity z_i drawn from a distribution with density $p(z)$ and support $[z, \bar{z}]$. Firms produce output $y_i = z_i^{1-\alpha} k_i^\alpha$. Suppose that a social planner wants to maximize aggregate output $Y = \int_0^M y_i d_i$ subject to $\int_0^M k_i d_i = \bar{K}$. Then at the optimum, aggregate output is given by $Y = ZK^\alpha M^{1-\alpha}$, where

$$Z = \left[\int_{\underline{z}}^{\bar{z}} z p(z) dz \right]^{1-\alpha}$$

Proof of proposition 9. We can write the maximization problem as

$$\max_{(k_i)_{i=0}^M} \left\{ \int_0^M z_i^{1-\alpha} k_i^\alpha \, \mathbf{d}i \right\} \quad \text{subject to} \quad \int_0^M k_i \, \mathbf{d}i = \bar{K}$$

The first-order condition for k_i is

$$\alpha z_i^{1-\alpha} k_i^{\alpha-1}, \forall i = \lambda$$

where λ is the multiplier on the resource constraint. Then

$$\alpha z_i^{1-\alpha} k_i^{\alpha-1} = \alpha z_j^{1-\alpha} k_j^{\alpha-1}, \forall i, j$$

Rearrange:

$$k_i = \left(\frac{z_i}{z_j} \right) k_j$$

Integrate with respect to j :

$$k_i = \int_0^M \left(\frac{z_i}{z_j} \right) k_j \, \mathbf{d}j = \left(\frac{z_i}{\int_0^M z_j \, \mathbf{d}j} \right) \bar{K}$$

Hence

$$\begin{aligned} Y &= \int_0^M y_i \, \mathbf{d}i \\ &= \int_0^M z_i^{1-\alpha} k_i^\alpha \, \mathbf{d}i \\ &= \int_0^M z_i^{1-\alpha} \left[\left(\frac{z_i}{\int_0^M z_j \, \mathbf{d}j} \right) \bar{K} \right]^\alpha \, \mathbf{d}i \\ &= \left[\int_0^M z_i \, \mathbf{d}i \right] \left[\int_0^M z_j \, \mathbf{d}j \right]^{-\alpha} \bar{K}^\alpha \\ &= \bar{K}^\alpha \left[\int_0^M z_i \, \mathbf{d}i \right]^{1-\alpha} \end{aligned}$$

Now let's use a standard trick used in models of monopolistic competition. Let P denote the CDF associated with the density p . The law of large numbers implies that for each z there are $MP(z)$ firms that draw productivities less than or equal to z . The FOC for k_i implies that all firms with the same z_i receive the same allocations. As a consequence we can transform the last equation above, taking the integral with respect to z instead of i :

$$Y = \bar{K}^\alpha \left[\int_{\underline{z}}^{\bar{z}} z M p(z) \, \mathbf{d}z \right]^{1-\alpha}$$

Slightly rearranging, we have

$$Y = Z\bar{K}^\alpha M^{1-\alpha}$$

where

$$Z = \left[\int_{\underline{z}}^{\bar{z}} zp(z) dz \right]^{1-\alpha}$$

Hence output is a function of aggregate capital, the mass of firms, and the average productivity. Working in the other direction, if we observe Y , K and M but not $p(\cdot)$, we can infer that average productivity Z (i.e. TFP) is decreasing in M for fixed Y and K . In other words, if there exist two economies which produce the same output Y with the same K , but economy 1 has half the mass of firms that economy 2 has, then firms in economy 1 must be more productive on average. \square

Lemma 2. For fixed k , $\omega([r^*, \infty), k; c)$ is bounded.

Proof of lemma 2. Clearly, $F(\underline{a}(k, r)|c)$ is bounded between 0 and 1, so for $\omega(\cdot, k; c)$ to be unbounded it must be that $(1+r)[1-F(\underline{a}(k, r)|c)]$ diverges as r goes to infinity. The first term $(1+r)$ diverges while the second goes to zero so we cannot use the product rule. Let $f(r) = 1/(1+r)$ and $g(r) = [1-F(\underline{a}(k, r)|c)]$. Then

$$\lim_{r \rightarrow \infty} \frac{f'(r)}{g'(r)} = \lim_{r \rightarrow \infty} \frac{f(\underline{a}(k, r)|c)}{(1+r)^2}$$

We therefore have $\lim_{r \rightarrow \infty} f(r) = \lim_{r \rightarrow \infty} g(r) = 0$ and $\lim_{r \rightarrow \infty} f(r)/g(r) = 0$, so by L'Hôpital's rule $\lim_{r \rightarrow \infty} f(r)/g(r) = \lim_{r \rightarrow \infty} (1+r)[1-F(\underline{a}(k, r)|c)] = 0$ as well. Hence $\omega([r^*, \infty), k; c)$ is bounded. \square

Lemma 3. For fixed k , there exists $\tilde{r} \geq r^*$ such that $\omega(k, \cdot; c)$ is increasing on $[r^*, \tilde{r}]$ and decreasing on $[\tilde{r}, \infty)$.

Proof of lemma 3. The partial derivative of α with respect to r is

$$\alpha_r(k, r; c) = \frac{\partial \omega(k, r; c)}{\partial r} = 1 - F(\underline{a}(k, r)|c) + \left(\frac{\phi}{1+r} - 1 \right) f(\underline{a}(k, r)|c)$$

The first part $1 - F(\underline{a}(k, r)|c)$ is always positive but and decreasing while the second part $\left(\frac{\phi}{1+r} - 1 \right) f(\underline{a}(k, r)|c)$ is always negative. I will show that if $\alpha_r(k, r; c) \geq 0$ then it is strictly decreasing, and if $\alpha_r(k, r; c) < 0$ then for all $r' \geq r$ $\alpha_r(k, r'; c) < 0$ as well. The desired result will then follow. \square

Proof of proposition 2. For part (i), we cannot use the theorem of the maximum directly because $[r^*, \infty)$, the domain of $\omega(\cdot, k; c)$, is not compact. However, we can still make use of the theorem indirectly. Take any function $r_n(k; c)$ such that $\lim_{n \rightarrow \infty} r_n(k; c) = \infty$ for each k and $\omega(r_n(k; c); c)$ converges uniformly to $-(1 +$

r^*). Now define $\bar{\omega}_n(k; c) = \max_{r \in [r^*, r_n(k; c)]} \omega(r, k, c)$. Applying the theorem of the maximum we find that $\bar{\omega}_n(k; c)$ is continuous for each n . The choice of $r_n(k; c)$, $\omega(r_n(k; c); c)$ immediately implies that $\bar{\omega}_n$ converges uniformly to $\bar{\omega}$ and hence $\bar{\omega}$ is continuous.

For part (ii), suppose $\exists k, k'$ such that $k' > k$ but $\bar{\omega}(k'; c) < \bar{\omega}(k; c)$. We have $\bar{a}(k, r)$ is strictly decreasing in k for fixed r , so $\bar{a}(k', \bar{r}(k; c)) < \bar{a}(k, \bar{r}(k; c))$, which in turn means that $F(\bar{a}(k', \bar{r}(k; c))|c) > F(\bar{a}(k, \bar{r}(k; c))|c)$. This implies that $\omega(k', \bar{r}(k; c); c) > \omega(k, \bar{r}(k; c); c)$. But it must be that $\bar{\omega}(k'; c) \geq \omega(k', r; c)$ for all r . So we have a contradiction.

Parts (iii) and (iv) are straightforward. For (iii), suppose $\exists A \in \mathbb{R}$ such that $\bar{\omega}(k; c) \leq A$ for all k . Note that $\lim_{k \rightarrow 0} \underline{a}(k, r) = 0$ for all r , so $\lim_{k \rightarrow 0} F(\underline{a}(k, r)|c) = 0$ for all r , which in turn means $\lim_{k \rightarrow 0} \omega(k, r; c) = r - r^*$ for all r . Hence $\exists k$ sufficiently small that $\omega(k, 2A + r^*; c) > A$. For such k it must be that $\bar{\omega}(k; c) \geq \omega(k, 2A + r^*; c) > A$. Contradiction. For (iv), note that $\lim_{k \rightarrow \infty} \omega(k, r^*; c) = \phi - (1 + r^*)$. From lemma 3 we know that there exists some sufficiently large k' such that for all $k \geq k'$, $\omega(k, r; c)$ is strictly decreasing in r for all $r \geq r^*$. Hence $\lim_{k \rightarrow \infty} \bar{r}(k; c) = r^*$. This means $\lim_{k \rightarrow \infty} \bar{\omega}(k; c) = (1 - \phi) - (1 + r^*)$. \square

Proof of proposition 3. The first part follows immediately from proposition 2 and the intermediate value theorem. For the second part, suppose there exist c, c' such that $c > c'$ but $\bar{k}(c) \leq \bar{k}(c')$. By definition of \bar{k} we then have $\bar{\omega}(\bar{k}(c); c) = \bar{\omega}(\bar{k}(c); c') = 0$. We know from proposition 1 that $F(\underline{a}(\bar{k}(c), r)|c) \leq F(\underline{a}(\bar{k}(c'), r)|c)$ for all r , so $\omega(\bar{k}(c), r; c) \geq \omega(\bar{k}(c'), r; c)$ for all r . This implies $\bar{\omega}(\bar{k}(c); c) \geq \bar{\omega}(\bar{k}(c'); c)$. We also know that $F(\underline{a}(\bar{k}(c'), r)|c) < F(\underline{a}(\bar{k}(c'), r)|c')$ for all r , so $\omega(\bar{k}(c'); c) > \omega(\bar{k}(c'); c')$ for all r , which implies $\bar{\omega}(\bar{k}(c'); c) > \bar{\omega}(\bar{k}(c'); c')$. Then we have

$$\bar{\omega}(\bar{k}(c); c) \geq \bar{\omega}(\bar{k}(c'); c) > \omega(\bar{k}(c'); c') = \bar{\omega}(\bar{k}(c); c)$$

This is a contradiction, so it must be that $\bar{k}(c) > \bar{k}(c')$. \square

Proof of proposition 4. Fix c . Suppose $\exists k, k'$ such that $k' > k$ but $r(k', c) \leq r(k, c)$. By definition of $r(\cdot, c)$, it must be that $\omega(k, r(k, c); c) = \omega(k', r(k', c); c) = 0$. Since $\underline{a}(\cdot, r)$ is strictly increasing for fixed r , $F(\underline{a}(\cdot, r)|c)$ is strictly increasing for fixed r . Hence $\omega(\cdot, r; c)$ is strictly decreasing for fixed r . Competition among lenders means that $r(k, c) \leq \bar{r}(k; c)$, and since \bar{r} is a maximizer 3 implies that $\omega(k, r; c)$ is increasing in r for all $r \leq \bar{r}(k; c)$. Then $\omega(k', r(k, c); c) \geq \omega(k', r(k', c); c)$. And since $\omega(\cdot, r(k, c); c)$ is strictly decreasing, $\omega(k, r(k, c); c) > \omega(k', r(k, c); c)$. Thus we have

$$\omega(k, r(k, c); c) > \omega(k', r(k, c); c) \geq \omega(k', r(k', c); c) = \omega(k, r(k, c); c)$$

This is a contradiction, so it must be that $r(k', c) > r(k, c)$.

Now fix k . Suppose c, c' such that $c > c'$ but $r(k, c) \geq r(k, c')$. By definition of $r(\cdot, c)$, it must be that $\omega(k, r(k, c); c) = \omega(k, r(k, c'); c') = 0$. We know from proposition 1 that $F(\underline{a}(k, r(k, c'))|c') > F(\underline{a}(k, r(k, c'))|c)$, so $\omega(k, r(k, c'); c') < \omega(k, r(k, c'); c)$. By a similar argument to the one used above, it must be that $\omega(k, r; c)$ is increasing in r for all $r \leq r(k, c)$, which implies that $\omega(k, r(k, c); c) \geq \omega(k, r(k, c'); c)$. Thus we have

$$\omega(k, r(k, c'); c') < \omega(k, r(k, c'); c) \leq \omega(k, r(k, c); c) = \omega(k, r(k, c'); c')$$

This is a contradiction, so it must be that $r(k, c) < r(k, c')$. □

Proof of proposition 5. These proofs are virtually identical to the proof of proposition 3. □

Lemma 4. *The derivative of the interest rate schedule can be expressed as $r_k(k, c) = \rho(k, c)/k$, where*

$$\rho(k, c) = -(1 - \alpha) \left[[1 - F(\underline{a}(k, r(k, c))|c)] + \frac{\phi - 1 - r(k, c)}{1 + r(k, c)} f(\underline{a}(k, r(k, c))|c) \right]^{-1} (\phi - 1 - r(k, c)) f(\underline{a}(k, r(k, c))|c)$$

Proof of lemma 4. This is an application of the implicit function theorem. The interest rate schedule requires that $\omega(k, r(k, c); c) = 0$ for all k . Fix (k_0, r_0) such that $k_0 < \bar{k}(c)$ and $\omega(k_0, r_0; c) = 0$. From the definition of $\bar{k}(c)$ it follows that $\bar{\omega}(k_0; c) > 0$. This means that any r_0 such that $\omega(k_0, r_0; c) = 0$ cannot be the maximizer associated with $\bar{\omega}(k_0; c)$. Therefore $\partial\omega(k_0, r_0; c)/\partial r \neq 0$ for all $k_0 < \bar{k}(c)$ and r_0 such that $\omega(k_0, r_0; c) = 0$. In other words, $\partial\omega(k_0, r_0; c)/\partial r$ is invertible for all such (k_0, r_0) .

Then the implicit function theorem implies that there exist open sets $U \ni k_0$ and $V \ni r_0$ and a continuously differentiable function $r(\cdot, c) : U \rightarrow V$ (i.e. the interest rate schedule) such that $\omega(k, r(k, c); c) = 0$ for all $(k, r) \in U \times V$ and

$$\frac{\partial r(k, c)}{\partial k} = - \left(\frac{\partial\omega(k, r(k, c); c)}{\partial r} \right)^{-1} \frac{\partial\omega(k, r(k, c); c)}{\partial k}.$$

We have

$$\begin{aligned} \frac{\partial\omega(k, r; c)}{\partial k} &= (\phi - 1 - r) f(\underline{a}(k, r)|c) \frac{\partial\underline{a}(k, r)}{\partial k} \\ \frac{\partial\omega(k, r; c)}{\partial r} &= [1 - F(\underline{a}(k, r)|c)] + (\phi - 1 - r) f(\underline{a}(k, r)|c) \frac{\partial\underline{a}(k, r)}{\partial r} \end{aligned}$$

Taking derivatives of \underline{a} we see that

$$\begin{aligned}\frac{\partial \omega(k, r; c)}{\partial k} &= (\phi - 1 - r) f(\underline{a}(k, r) | c) \frac{1 - \alpha}{k} \\ \frac{\partial \omega(k, r; c)}{\partial r} &= [1 - F(\underline{a}(k, r) | c)] + \frac{\phi - 1 - r}{1 + r} f(\underline{a}(k, r) | c)\end{aligned}$$

This means that

$$r_k(k, c) = - \left[[1 - F(\underline{a}(k, r(k, c)) | c)] + \frac{\phi - 1 - r(k, c)}{1 + r(k, c)} f(\underline{a}(k, r(k, c)) | c) \right]^{-1} (\phi - 1 - r(k, c)) f(\underline{a}(k, r(k, c)) | c) \frac{1 - \alpha}{k}$$

Define

$$\rho(k, c) = -(1 - \alpha) \left[[1 - F(\underline{a}(k, r(k, c)) | c)] + \frac{\phi - 1 - r(k, c)}{1 + r(k, c)} f(\underline{a}(k, r(k, c)) | c) \right]^{-1} (\phi - 1 - r(k, c)) f(\underline{a}(k, r(k, c)) | c)$$

Then

$$r_k(k, c) = \frac{\rho(k, c)}{k}$$

as claimed. □

Lemma 5. *The default probabilities $F(\underline{a}(k, r) | c)$, the interest rate schedule $r(k, c)$ and the function $\rho(k, c)$ above can be expressed as functions of $\xi = (1 - \alpha) \log(k) - \mu_{a|c}$.*

Proof of lemma 5. Recall that the cutoff productivity is

$$\underline{a}(k, r) = \log \left((1 + r) p k^{1 - \alpha} \right) = \log(1 + r) + \log(p) + (1 - \alpha) \log(k).$$

Define the re-centered productivity $\tilde{a}_i = a_i - \mu_{a|c}$. Clearly $\tilde{a}_i | c \sim \mathcal{N}(0, \sigma_{a|c}^2)$, and note that $\sigma_{a|c}^2$ does not depend on c . Let K denote the CDF associated with this re-centered distribution. We can write the no-arbitrage condition as

$$1 + r^* = (1 + r(k, c)) \left[1 - K(\underline{a}(k, r(k, c)) - \mu_{a|c}) \right] + \phi K(\underline{a}(k, r(k, c)) - \mu_{a|c})$$

Use the formula for the cutoff:

$$\begin{aligned}1 + r^* &= (1 + r(k, c)) \left[1 - K(\log(1 + r(k, c)) + \log(p) + (1 - \alpha) \log(k) - \mu_{a|c}) \right] \\ &\quad + \phi K(\log(1 + r(k, c)) + \log(p) + (1 - \alpha) \log(k) - \mu_{a|c})\end{aligned}$$

Use the definition of ξ_i :

$$1 + r^* = (1 + r(k, c)) [1 - K(\log(1 + r(k, c)) + \log(p) + \xi)] + \phi K(\log(1 + r(k, c)) + \log(p) + \xi)$$

Clearly $r(k, c)$ depends only on ξ . Similar logic applies to $\rho(k, c)$ and the default probability $K(\log(1 + r(k, c)) + \log(p) + \xi)$. Hence we can write $r(k, c) = r(\xi)$, $\rho(k, c) = \rho(\xi)$ and $K(\log(1 + r(\xi)) + \log(p) + \xi)$. \square

Proof of proposition 6. The proof follows from the firm's first-order condition and propositions 4 and 5. \square

Proof of proposition 7. Similar logic applies here. \square

Proof of proposition 8. Use the fact that $\mu_{a|c} = \mu_{a|b}$ when $\sigma_\eta = 0$ in the first-order condition (13), the rule for partial expectations with lognormal distributions, and lemmas 4 and 5:

$$\alpha \exp \left[-\xi + \frac{\sigma_{a|b}^2}{2} \right] \left[1 - K(\log(1 + r(\xi)) + \log(p) + \xi - \sigma_{a|b}^2) \right] = [(1 + r(\xi))p + \rho(\xi)] [1 - K(\log(1 + r(\xi)) + \log(p) + \xi)]$$

It's immediately clear that all firms choose the same ξ , and thus the same expected marginal products. When $\mu_{a|c}$ is not equal to $\mu_{a|b}$, this is no longer the case. \square

Proposition 10. Suppose $\sigma_\eta = 0$. Consider a social planner who observes all firms' signals b and chooses capital allocations and default cutoff rules for each firm to maximize total output. The planner's problem

$$\max_{k(b), \underline{a}(b)} \left\{ \int_{-\infty}^{\infty} \int_{\underline{a}(b)}^{\infty} e^a k(b)^\alpha f(a|b) p(b) da db \right\} \quad \text{subject to} \quad \int_{-\infty}^{\infty} \int_{\underline{a}(b)}^{\infty} k(b) f(a|b) p(b) da db$$

yields a solution that satisfies

$$\alpha k(b)^{\alpha-1} \int_{\underline{a}(b)}^{\infty} e^a f(a|b) da = \alpha k(b')^{\alpha-1} \int_{\underline{a}(b')}^{\infty} e^a f(a|b') da, \quad \forall b, b'$$

In other words, all firms have the same expected marginal product.

Proof of proposition 10. The first order condition for $k(b)$ is

$$\alpha k(b)^{\alpha-1} \int_{\underline{a}(b)}^{\infty} e^a f(a|b) da = \lambda [1 - F(\underline{a}(b)|b)]$$

where λ is the multiplier on the resource constraint. The first-order condition for $\underline{a}(b)$ is

$$\underline{a}(b) = \log(\lambda) + (1 - \alpha) \log(k(b))$$

The rest of the proof follows from similar arguments to those used in the proofs of lemma 5 and proposition 8. □